ISSN 0027-1330

Moscow University

Mechanics Bulletin

(Vestnik Moskovskogo Universiteta. Mekhanika)

工業大学
7. 7. 20
附属図書館

Vol. 49, No. 5

ALLERTON PRESS, INC.

Moscow University Mechanics Bulletin
Vol. 49, No. 4, pp. 71-74, 1994

Vestnik Moskovskogo Universiteta. Mekhanika
UDC 539.3:534.1

MULTIPLE-FIELD APPROACH IN MODELING STABILITY OF A CYLINDRICAL SHELL STIFFENED WITH FRAMES AT EXTERNAL PRESSURE

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The derivation of one-field stability equations is presented for a cylindrical shell regularly stiffened with frames and loaded with an external pressure. Stability equations for two-field models are derived in a similar manner by introducing a macrocell.

The multiple-field approach is efficient in constructing theories of elasticity of complex lattices, in constructing models of continuous media with taking into account the macrostructure [1], and in modeling the stability of discrete systems [2, 3]. In this paper we consider the application of the multiple-field approach for modeling the stability of a continual system and derive two-field stability equations for a cylindrical shell regularly stiffened with elastic frames and loaded with a uniform external hydrostatic pressure p.

1. MAIN RELATIONS FOR STIFFENED SHELL

We assume [4] that the behavior of the shell is described by equations of the quasimembrane theory, the width of the frames is negligibly small compared with the distance between them, the centers of gravity of their cross-sections lie in the median surface of the shell, the axes are inextensible, and only the bending rigidity of the frames in their plane is taken into account.

The shell is characterized by the following parameters: R is the radius, h is the thickness of the shell, a is the distance between the frames, EJ is the bending rigidity of the frame, B_x is the rigidity of the casing for extension in the axial direction, and D_φ is the bending rigidity of the shell in the circumferential direction.

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The loss of stability of the m-th span is described by the equation

$$\hat{L}\Phi_m = 0, \quad \hat{L} = B_x \frac{\partial^4}{\partial x^4} + \frac{D_\varphi}{R^6}\frac{\partial^4}{\partial \varphi^4}\left(\frac{\partial^4}{\partial \varphi^4} + 2\frac{\partial^2}{\partial \varphi^2} + 1\right) + \frac{p}{R^3}\frac{\partial^4}{\partial \varphi^4}\left(\frac{\partial^2}{\partial \varphi^2} + 1\right), \tag{1}$$

where $\Phi_m(x, \varphi)$ is the displacement function defined within the span ($0 \leqslant x \leqslant a$). The relations for the axial displacements u, circumferential displacements v, axial forces T_x, and shear forces S, expressed via the displacement function, have the form

$$u = \frac{\partial \Phi}{\partial x}, \quad v = -\frac{1}{R}\frac{\partial \Phi}{\partial \varphi}, \quad T_x = B_x \frac{\partial^2 \Phi}{\partial x^2}, \quad \frac{\partial S}{\partial \varphi} = -B_x R \frac{\partial^3 \Phi}{\partial x^3}.$$

For the circumferential displacements V of the points of the m-th frame we have the equation

$$EJ\frac{\partial^2}{\partial \varphi^2}\left(\frac{\partial^4 V}{\partial \varphi^4} + 2\frac{\partial^2 V}{\partial \varphi^2} + V\right) = R^4(S_{m-1}(a) - S_m(0)).$$

Solution to Eq. (1) is sought in the form $\Phi_m(x, \varphi) = X(x)\sin(n\varphi)$. Transformations with the use of the joint conditions (the equality of the circumferential and axial displacements, and axial forces, in passing through the frame, the equality of the circumferential displacements of the points of the frame axis and of the casing) give the following equation for the values $\Phi_m^0 = X_m(0)e^{in\varphi}$:

$$\Phi_{m+2}^0 + 2\Phi_m^0 + \Phi_{m-2}^0 + g(\lambda\bar{a})\Phi_m^0 - f(\lambda\bar{a})(\Phi_{m+1}^0 + \Phi_{m-1}^0) = 0, \tag{2}$$

where

$$f(\lambda\bar{a}) = 2[\cos(\lambda\bar{a}) + \cosh(\lambda\bar{a})] + c(\bar{a}\lambda)^{-3}[\sin(\lambda\bar{a}) - \sinh(\lambda\bar{a})]/2,$$

$$g(\lambda\bar{a}) = 4\cosh(\lambda\bar{a})\cos(\lambda\bar{a}) + c(\bar{a}\lambda)^{-3}[\cosh(\lambda\bar{a})\sin(\lambda\bar{a}) - \cos(\lambda\bar{a})\sinh(\lambda\bar{a})],$$

$$\bar{a} = \frac{a}{R}, \quad c = \frac{EJ\bar{a}^3}{B_x R^3}n^4(n^2-1)^2, \quad \lambda^4 = \frac{R}{B_x}n^4(n^2-1)\left(p - \frac{D_\varphi}{R^3}(n^2-1)\right).$$

Substituting $\Phi_m^0 = \Phi^0 e^{ikma}$ into (2) we obtain the characteristic equation

$$4\cos^2(ka) - f(\lambda\bar{a})2\cos(ka) + g(\lambda\bar{a}) = 0. \tag{3}$$

2. ONE-FIELD MODELS

To construct the one-field model, we introduce the fur ction $\Phi(x, \varphi)$ defined along the entire axis of the shell, such that in the places where the frames are disposed the relation

$$\Phi(x, \varphi)|_{x=ma} = \Phi_m(0, \varphi)$$

holds. Taking the Taylor-series expansion of the functions $g(\lambda\bar{a})$ and $f(\lambda\bar{a})$ with respect to λ in relation (3) and replacing n by $i\partial()/\partial\varphi$ give the differential (with respect to φ)-difference (with respect to x) one-field stability equation

$$\hat{L}\Phi = 0, \quad \hat{L} = [\nabla_{-2a}\nabla_{+2a} + 4] + \hat{G}_\infty - \hat{F}_\infty[\nabla_{-a}\nabla_{+a} + 2], \tag{4}$$

where

$$\nabla_{\pm\xi}\Phi(x, \varphi) = \pm[\Phi(x \pm \xi, \varphi) - \Phi(x, \varphi)],$$

$$\hat{G}_s = \sum_{r=0}^{s} \left(\frac{(-1)^r 4^{r+1}}{(4r)!} + \frac{(-1)^r 4^{r+1}}{(4r+3)!} \hat{L}_c \right) (\bar{a}\hat{\Lambda})^{4r},$$

$$\hat{F}_s = \sum_{r=0}^{s} \left(\frac{4}{(4r)!} + \frac{-1}{(4r+3)!} \hat{L}_c \right) (\bar{a}\hat{\Lambda})^{4r},$$

$$\hat{L}_c = \frac{EJ\bar{a}^3}{B_x R^3} \frac{\partial^4}{\partial\varphi^4} \left(\frac{\partial^2}{\partial\varphi^2} + 1 \right)^2,$$

$$\hat{\Lambda}^4 = -\frac{R}{B_x} \frac{\partial^4}{\partial\varphi^4} \left(\frac{\partial^2}{\partial\varphi^2} + 1 \right) \left(p + \frac{D_\varphi}{R^3} \left(\frac{\partial^2}{\partial\varphi^2} + 1 \right) \right).$$

Taking the Taylor-series expansion of the function Φ in (4) with respect to x to the fourth-order derivatives, we shall obtain the one-field stability equation, differential with respect to x and φ:

$$\hat{L}\Phi = 0, \quad \hat{L} = \hat{K}_1 + \hat{G}_s - \hat{F}_s\hat{K}_2, \quad s = \infty, \tag{5}$$

where $\hat{K}_1 = \frac{4}{3}a^4\frac{\partial^4}{\partial x^4} + 4a^2\frac{\partial^2}{\partial x^2} + 4$ and $\hat{K}_2 = \frac{1}{12}a^4\frac{\partial^4}{\partial x^4} + a^2\frac{\partial^2}{\partial x^2} + 2$. The substitution of $\Phi(x, \varphi) = Xe^{ikx}\sin(n\varphi)$ gives the characteristic equation

$$K_1(ak) + g(\lambda\bar{a}) - f(\lambda\bar{a})K_2(ak) = 0, \tag{6}$$

where

$$K_1(ak) = 4(ak)^4/3 - 4(ak)^2 + 4, \quad K_2(ak) = (ak)^4/12 - (ak)^2 + 2.$$

This equation is obtained from exact equation (3) as the result of taking the Taylor-series expansion of the functions $4\cos^2(ka)$ and $2\cos(ka)$ to the fourth-order derivatives at $k = 0$.

3. TWO-FIELD MODELS

To construct the two-field model, we take a macrocell comprising two elementary periods as the period. Using additional indices 0 and 1 for denoting the values of the displacement functions in the beginning of even and odd spans, we shall rewrite Eq. (2) as system (7):

$$\begin{aligned}&\Phi^0_{0,n+2}+2\Phi^0_{0,n}+\Phi^0_{0,n-2}+g(\lambda\bar{a})\Phi^0_{0,n}-f(\lambda\bar{a})(\Phi^0_{1,n+1}+\Phi^0_{1,n-1})=0,\\&-f(\lambda\bar{a})(\Phi^0_{0,n+2}+\Phi^0_{0,n})+\Phi^0_{1,n+3}+2\Phi^0_{1,n+1}+\Phi^0_{1,n-1}+g(\lambda\bar{a})\Phi^0_{1,n+1}=0,\quad n=2m.\end{aligned}\tag{7}$$

In the same way as we derived Eq. (4) from Eq. (2), introduce two functions $\Phi_r(x,\varphi)$ $(r=0,1)$ such that

$$\Phi_r(x,\varphi)|_{x=(2m+r)a}=\Phi^0_{2m+r}(0,\varphi),\quad r=0,1,$$

and set up a system of differential-difference equations using Eqs. (7). Taking the Taylor-series expansion of the functions Φ_0 and Φ_1 with respect to x to the fourth-order derivatives, we shall obtain the differential stability equations for the two-field model that are approximate with respect to x:

$$\begin{aligned}&[\hat{K}_1+\hat{G}_s]\Phi_0-\hat{F}_s\hat{K}_2\Phi_1=0,\\&-\hat{F}_s\hat{K}_2\Phi_0+[\hat{K}_1+\hat{G}_s]\Phi_1=0,\quad s=\infty.\end{aligned}\tag{8}$$

The corresponding characteristic equation has the form $D_-D_+=0$, where

$$D_\pm=K_1(ak)-g(\lambda\bar{a})\pm f(\lambda\bar{a})K_2(ak).$$

The equation $D_-=0$ coincides with Eq. (6). The equation $D_+=0$ can be obtained from the exact equation, if we replace k by $\pi/a-k$ in it and take the Taylor-series expansion of the functions $4\cos^2(ka)$ and $-2\cos(ka)$ to the fourth-order derivatives at $k=0$. In other words, the two-field model, coinciding with the one-field model at small k, refines it at k close to π/a. To this value of k there corresponds the pressure value in the case of the local loss of stability.

4. APPLIED MODELS

Eliminating all the derivatives with respect to x and φ of order higher than the eighth in Eq. (5) yields stability equation (1) of the quasimembrane theory for the shell with the bending rigidity $D_\varphi+EJ/a$, used in the constructively anisotropic approach.

Equations (5) and (8) for finite s's give one-field and two-field equations which are applicable for finding s minimum critical forces and convenient for analysis.

At $s=1$ the relations of the one- and two-field models of stability, approximate with respect to x and φ, are formed from (5) and (8), which give, respectively, formulas (9) and (10) for finding the critical pressure value by minimization with respect to n:

$$p = \lambda^4 B_x \frac{1}{Rn^4(n^2-1)} + \frac{D_\varphi}{R^3}(n^2-1), \quad (\bar{a}\lambda)^4 = a_1/a_1, \tag{9}$$

$$a_0 = K_1(ak) + 4 + \frac{2}{3}c - \left(4 - \frac{c}{6}\right) K_2(ak),$$

$$a_1 = \left(\frac{2}{3} + \frac{16c}{7!}\right) + \left(\frac{1}{6} - \frac{c}{7!}\right) K_2(ak);$$

$$p^{\pm} = \lambda_{\pm}^4 B_x \frac{1}{Rn^4(n^2-1)} + \frac{D_\beta}{R^3}(n^2-1), \quad \bar{a}^4\lambda_{\pm}^4 = a_0^{\pm}/a_1^{\pm}, \tag{10}$$

$$a_0^{\pm} = K_1(ak) + 4 + \frac{2}{3}c - \left(4 - \frac{c}{6}\right)(\pm K_2(ak)),$$

$$a_1^{\pm} = \left(\frac{2}{3} + \frac{16c}{7!}\right) + \left(\frac{1}{6} - \frac{c}{7!}\right)(\pm K_2(ak)).$$

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10 December 1993