2. Multi-field modelling of Cosserat solids

2.1. Cosserat lattice. Discrete model. We consider a Cosserat lattice, i.e. a lattice whose deformations are described by displacements \( u \) and by rotation \( \theta \) of its elements. The elements are placed at the nodes of a square lattice as in Fig. 1(a).

The potential energy associated with the elastic connection of elements \( i,j \) has the following form:

\[
\Phi = \sum_{i,j} \left( 2\alpha \left( u_{ij} - u_{d} \right)^2 + \beta \left( \theta_{ij} - \theta_{d} \right)^2 \right)
\]

The expression for the kinetic energy of the system has the form:

\[
\mathcal{E} = \sum_{i,j} \left( \frac{1}{2} k u_{ij}^2 + \frac{1}{2} \lambda \left( \theta_{ij} \right)^2 \right)
\]

Discrete equations of motion are obtained by Lagrange's equations.

2.2. Single-field micro-model. In the micromodel it is assumed that deformations of a discrete element are described by the single vector functions \( u(x,y) = (u_x(x,y), u_y(x,y), u_z(x,y)) \), which has the same components of the vector of generalized displacements \( \mathbf{r}(x,y) = (r_x(x,y), r_y(x,y), r_z(x,y)) \) of the unit element. In this case the displacements are expressed as a function of coordinates by means of the equations of inter-element connections.

In the new co-ordinates \( O' \), discrete equation of motion has the form:

\[
\sum_{i,j} M_{ij} \left( \frac{1}{2} k' \left( u_{ij} - u_{ij}' \right) + \frac{1}{2} \lambda \left( \theta_{ij} - \theta_{ij}' \right) \right) = \frac{1}{2} \sum_{i,j} F_{ij} \left( u_{ij} - u_{ij}' \right)
\]

Discrete micro-model [3] includes derivation of the fourth order.

2.3. Hierarchical system of multi-field micropolar models. For deriving the \( N \)-field micropolar model we consider a basic micropolar, which belongs to \( N \)-dimensional system \([7, 13] \). Although, all elements of the lattice (Eq. 4) are identical, they are marked with different numbers (as examples are shown in Fig. 4b-c). We use the coefficients \( a_i \) and \( \theta_i \) which are the same for all micropolar displacements. Thus, for the particles marked by different numbers we obtain discrete equations of motions instead of using the single vector function. N-vector function \( \mathbf{L}(O') = (L_n(O'), L_y(O'), L_z(O')) \) are used in the N-field theory to describe the deformations and rotations of particles marked by numbers \( \nu \) for \( i, x, y, \), respectively. By using Taylor series expansions of displacements and rotations in the equations of the particles around the points at which the equations are written, we come to equations of N-field theory.

\[
L_n(O') = L_n(O) + \frac{1}{2} \frac{\partial L_n}{\partial u_n} \left( \frac{\partial u_n}{\partial x} \right) + \frac{1}{2} \frac{\partial L_n}{\partial \theta_n} \left( \frac{\partial \theta_n}{\partial x} \right)
\]

This equation can be obtained independently by using Taylor series expansions in Eq. (4).

Similarly, changing the variables and considering one-dimensional displacements, additional equation of 2D two-field model (model \( N = 2 \)) leads to the Cauchy\,—Steady, Cosserat\,—Steady, and Cosserat\,—Steady\,—Vlasov micro-models. For the \( N \)-field model \( \partial u_n / \partial x = \partial \theta_n / \partial x = 0 \) and \( \nu = 1, 2, \ldots, N \) is used to derive the model directly from Eq. (4).

In order to explain the notation and to illustrate the method of derivation of the multi-field models derived in this paper, we get the first field model directly from Eq. (4).

\[
\frac{\partial u_1}{\partial x} = \frac{1}{2} \frac{\partial L_1}{\partial u_1} \left( \frac{\partial u_1}{\partial x} \right) + \frac{1}{2} \frac{\partial L_1}{\partial \theta_1} \left( \frac{\partial \theta_1}{\partial x} \right)
\]

Although the unit cell in the problem under consideration consists of the single layer, we assume that a cell of periodicity consists of two layers and use the equations \( \mathbf{L}(O') \) and \( \mathbf{L}(O') \) for displacements of the layers within \( \mathbf{C} \) and \( \mathbf{C} \), respectively. Eq. (4) can then be rewritten in the form:

\[
\frac{\partial u_1}{\partial x} = \frac{1}{2} \frac{\partial L_1}{\partial u_1} \left( \frac{\partial u_1}{\partial x} \right) + \frac{1}{2} \frac{\partial L_1}{\partial \theta_1} \left( \frac{\partial \theta_1}{\partial x} \right)
\]

We use two functions \( u'(O') \) and \( \theta'(O') \) in order to describe deformations of odd and even layers. The Taylor series expansions of the disepnments in Eq. (5) and (6) up to fourth order terms around the points for which these equations were obtained, gives the system of coupled equations for the two-field model.

\[
\frac{\partial u_1}{\partial x} = \frac{1}{2} \frac{\partial L_1}{\partial u_1} \left( \frac{\partial u_1}{\partial x} \right) + \frac{1}{2} \frac{\partial L_1}{\partial \theta_1} \left( \frac{\partial \theta_1}{\partial x} \right)
\]

2.4. Plane wave solutions. The approximate analysis of the models. We compare models by using plane wave solutions:

\[
\mathbf{L}(O') = \sum_{k,m} \mathbf{L}_{k,m} e^{ikx} e^{imy}
\]

The dispersion curves of the convergent and higher-order gradient single-field models coincide with the dispersion curves of the discrete systems at the point \( (k, \ell, m, n) \), and approximate them around this point \([3]\). The higher-order gradient model improves the approximation at this point in comparison with the classical micropolar model. However, for short wavelength waves both single-field micropolar models produce results with an essential error.

2.5. One-dimensional solutions for this layer (structural interfaces). Also generalized models are more general and include conventional model, they are more complex and one of important questions in generalized continuum mechanics is the questions in the micromechanics, at the same time, model can be used, and should be used! We consider the one-dimensional deformations of a lattice placed between two rigid semi-infinite solids. (see Fig. 7) [7]. Assuming that the generalized deformations are constant for elements along the \( x \)-axis, the \( x \)-direction does not experience disepnment. In this case, the non-dimensionalized abbreviater equations \( \mathbf{L}_\nu = \mathbf{L}_\nu \) and \( \mathbf{L}_\nu = \mathbf{L}_\nu \) are decoupled, and we will consider the one-dimensional discrete model of the membrane with deformation by means of the vector of generalized displacements \( \mathbf{r}(x,y) = (r_x(x), r_y(x), r_z(x)) \) of the unit element. In this case the displacements are expressed as a function of coordinates by means of the equations of inter-element connections.

In the new co-ordinates \( O' \), discrete equation of motion has the form:

\[
\sum_{i,j} M_{ij} \left( \frac{1}{2} k' \left( u_{ij} - u_{ij}' \right) + \frac{1}{2} \lambda \left( \theta_{ij} - \theta_{ij}' \right) \right) = \frac{1}{2} \sum_{i,j} F_{ij} \left( u_{ij} - u_{ij}' \right)
\]

Correspondent one-dimensional solutions of the single-field higher-order gradient model have the form:

\[
\frac{\partial u_1}{\partial x} = \frac{1}{2} \frac{\partial L_1}{\partial u_1} \left( \frac{\partial u_1}{\partial x} \right) + \frac{1}{2} \frac{\partial L_1}{\partial \theta_1} \left( \frac{\partial \theta_1}{\partial x} \right)
\]

This equation can be obtained independently by using Taylor series expansions in Eq. (4).

3. Conclusion

The N-field theory is obtained as a continuum analog for the discrete model with a periodic cell consisting of \( N \) micropolar elements. It is shown that for \( N = 1 \) we get the well-known micropolar model. The \( N = 2 \) model is described with the help of two micropolar elements. By increasing \( N \), the model is described with \( N \) micropolar elements.

For the \( N = 3 \) model we obtain the following solutions for the equilibrium of the system, which is used to study the stability of the system.

\[
\frac{\partial u_1}{\partial x} = \frac{1}{2} \frac{\partial L_1}{\partial u_1} \left( \frac{\partial u_1}{\partial x} \right) + \frac{1}{2} \frac{\partial L_1}{\partial \theta_1} \left( \frac{\partial \theta_1}{\partial x} \right)
\]

\[
\frac{\partial u_2}{\partial x} = \frac{1}{2} \frac{\partial L_2}{\partial u_2} \left( \frac{\partial u_2}{\partial x} \right) + \frac{1}{2} \frac{\partial L_2}{\partial \theta_2} \left( \frac{\partial \theta_2}{\partial x} \right)
\]

\[
\frac{\partial u_3}{\partial x} = \frac{1}{2} \frac{\partial L_3}{\partial u_3} \left( \frac{\partial u_3}{\partial x} \right) + \frac{1}{2} \frac{\partial L_3}{\partial \theta_3} \left( \frac{\partial \theta_3}{\partial x} \right)
\]

Fig. 11. Amont's material.

Fig. 12. Material with chiral microstructure.