MULTI-FIELD MODELLING OF SHORT WAVELENGTH DEFORMATIONS FOR COSSERAT SOLIDS

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1. Introduction

1.1. Why continuum modelling? Field theories are effectively used for modelling structural systems. Some reasons are as follows: continuum models make it possible to define generalized macro-characteristics of systems; in some cases continuum models make it possible to find analytical solutions by using well-developed mathematical methods; in cases when analytical solutions cannot be found, one can use an effective numerical methods and packages based on artificial discretisation; field theories represent itself coupled set of interpenetrating theories.

There are however structural effects, which are not captured through classical continuum models. This may leads to essential errors in application. The study of such effects within the framework of the field theories require the development of generalized continuum models.

1.2. Generalised continuum modelling of cellular solids: three of possible approaches. Physical path to the development of generalized models consists in the analysis and evaluation of key physical hypotheses of existing models and further their rejection or generalization.

One can note the following possible ideas:

- Intra-cell approach: additional internal degrees of freedom for unit cell are considered. For example, in Cosserat and micropolar models rotational degrees of freedom of structural elements are taken into consideration in addition to displacements [1, 2].
- Higher derivatives of the fields are taken into account in higher-order gradient models.
- Macro-cell approach: by using a macro-cell, comprising of several elementary unit cells, and, accordingly, by increasing the number of vector fields in order to describe the deformations of the system we come to models of multi-field theory.

It is importunate to to note, that these hypotheses are mutually independent, complementary and can be used in various combinations. This note will be used in Sec. 2.3.

1.3. Where multi-field approach have been work? Some examples

- Stability problems: continuum modelling of loss of stability as for long- such for short-wavelength forms (Fig. 1).
- Short-wave deformations near boundaries, defects, localised forces (Fig. 2).
- Phase transitions: multi-field soliton dynamics (Fig. 3).

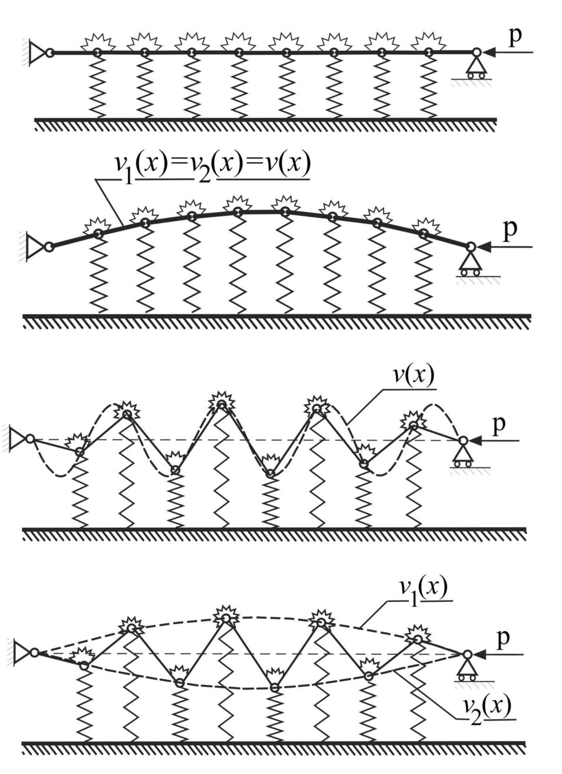


Fig. 1. As long- such sort-wavelength forms may be effectively described by slowly varying functions in two-field model, while single function described short-wave form should be highly varying.

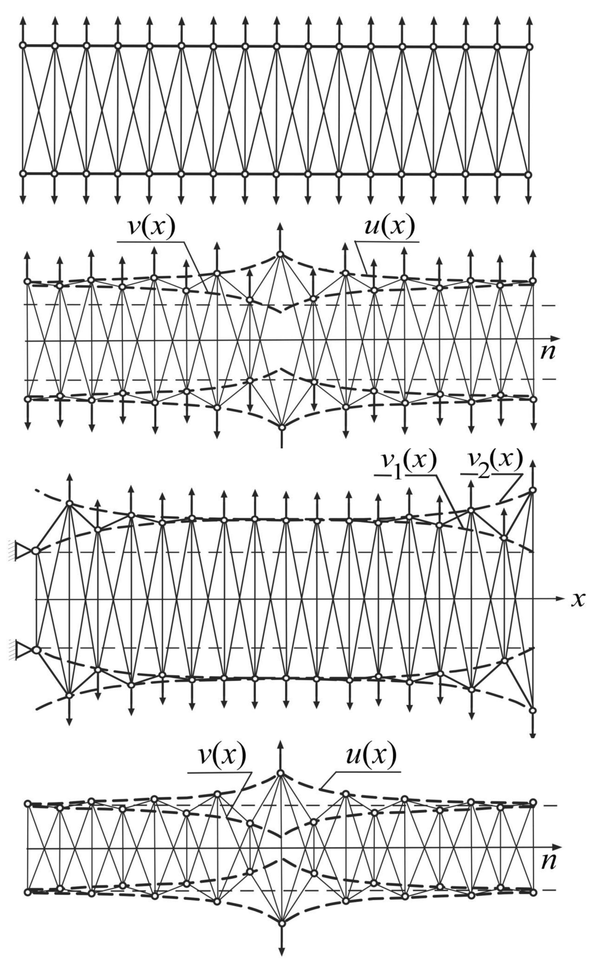


Fig 2.

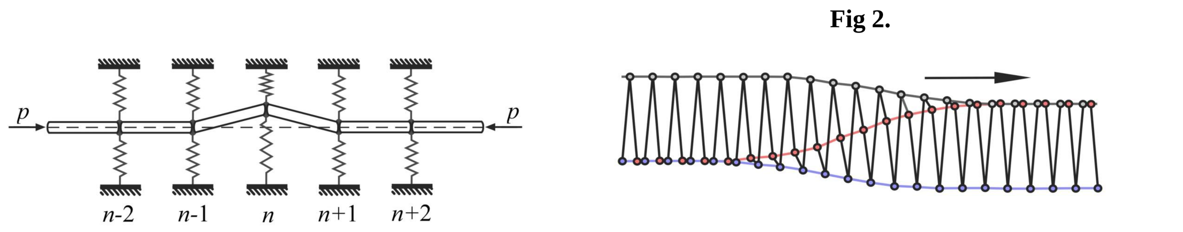


Fig. 3. A layer under internal pressure. Multi-field description of phase transition between two three-periodic static solutions.

1.4. Why multi-field approach works? From physical point of view, the multi-field theory is based on clear physical assumptions, which are generalization of the basic hypothesis.

Mathematically, Figs 1-3 demonstrate that both slowly and rapidly varying displacements are effectively described by two smooth field functions. This may help qualitatively understand why multi-field models with lower gradient terms provide good approximations for both slowly and rapidly varying displacements. Long-wavelengths deformations may be well described by using a slowly varying single function of generalized displacements but when we try to describe the short wavelengths deformations it should rapidly vary in the corresponding areas. For this reason, the single-field models are well for long-wavelength solutions but do not give good approximations for long-wavelength solutions or do not capture them at all.

2. Multi-field modelling Cosserat solids

2.1. Cosserat lattice. Discrete model. We consider a Cosserat lattice, i.e. a lattice whose deformations are described by displacements u_n, v_n, and by rotations φ_n of its elements. The elements are placed at the nodes of a square lattice as is shown in Fig. 1a.

The potential energy associated with the elastic connection of elements m, k has the following form

$$2E_{pot}^{k,m} = K_n^{k,m}(u_m-u_k)^2 + K_s^{k,m}\left[v_m - v_k - r_{k,m}\frac{\varphi_m+\varphi_k}{2}\right]^2 + G_r^{k,m}(\varphi_m-\varphi_k)^2, \tag{1}$$

The expression for the kinetic energy of elements has the form

$$E_{kin}^{k} = \frac{1}{2}M\dot{u}_k^2 + \frac{1}{2}M\dot{v}_k^2 + \frac{1}{2}I\dot{\varphi}_k^2. \tag{2}$$

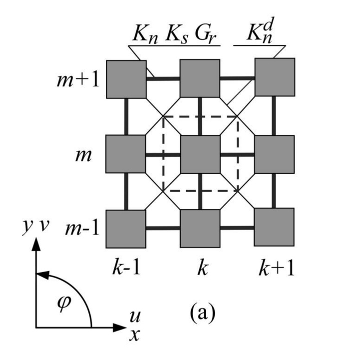
Discrete equations of motion are obtained by using Lagrange's equations.

2.2. Single-field micropolar model. In the micropolar model it is assumed that deformations of a discrete system can be described by using the single vector function $\{u(x,y,t), v(x,y,t), \varphi(x,y,t)\}$, which has the same components of the vector of generalized displacements $\{u_{k,m}(t), v_{k,m}(t), \varphi_{k,m}(t)\}$ of the unit cell. It is assumed that vector-function coincide with vector of displacements at nodes (kh, mh). The substitution $w(x\pm h, y\pm h)$ instead $w_{k\pm1,m\pm1}$ in the discrete equations and using Taylor series expansions

$$w(x\pm h, y\pm h, t) = e^{\pm h\partial_x \pm h\partial_y} w(x,y,t) = \sum_{r=0}^{N_x}\sum_{p=0}^{N_y} \frac{(\pm h)^r}{r!}\frac{(\pm h)^p}{p!}\frac{\partial^{r+p} w(x,y)}{\partial x^r \partial y^p} \tag{3}$$

gives a set of equations, which are differential with respect to spatial and temporal variables. Keeping derivatives up to the second order leads to the conventional single-field conventional model [3]. Higher order micropolar model [4] include derivatives of the fourth order.

2.3. Hierarchical system of multi-field micropolar models. For deriving the N-field model we consider as a basis a macrocell, which consists of N elementary cells [5, 7, 8]. Although, all elements of the lattice (Fig. 4a) are identical they are marked with different numbers (an examples are shown in Fig. 4b-e). We use the notations $u_{k,m}^{[n]}$, $v_{k,m}^{[n]}$, $\varphi_{k,m}^{[n]}$ with superscript n=1..N for the components of vector of generalised displacements. Then, for the particles marked by different numbers we obtain discrete equations of motion. Instead of using the single vector function, N vector functions $\{u^{[n]}(x,y,t), v^{[n]}(x,y,t), \varphi^{[n]}(x,y,t)\}$ are used in the N-field theory to describe the displacements and rotations of particles marked by numbers n=1..N, respectively. By using Taylor series expansions of displacements and rotations in the discrete equations around the points at which the equations are written, we come to equations of N-field theory.



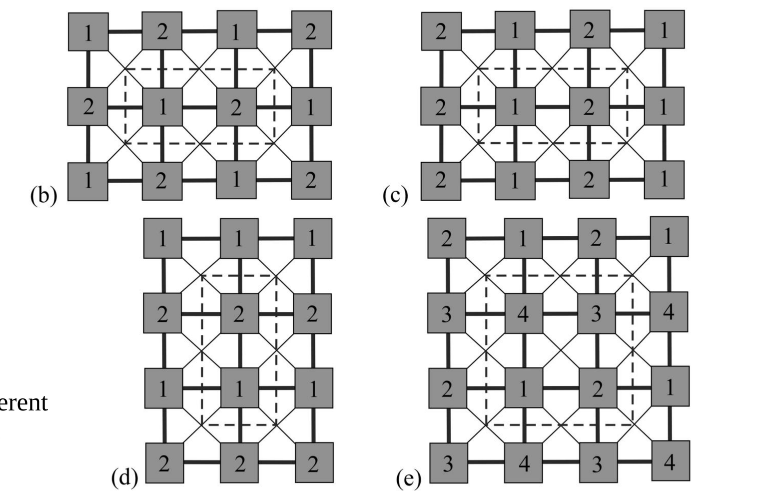


Fig. 4. Square lattice model. Different representation of square lattice as interpenetrating lattices

2.3.1. Two-field micropolar models. By using the procedure described above, we derive three types of the two-field models, N=2, corresponding to macro-cells presented in Fig. 4b-d [5]. In the linear case, the derived systems of six equations can be split in two uncoupled systems. One of them is the system of the equations of the micropolar theory, and, therefore, all models possess properties of conventional micropolar model. The second system varies with the model. Its meaning will be clarified by the following analysis in Sec. 2.5.

2.3.2. Four-field micropolar model. The four-field model correspond to macro-cell shown in Fig. 4e. It consist of twelve equations and combines the conventional Cosserat model and all two-field models obtained in Sec. 2.3.1 into one [5].

2.3.3. Higher-order gradient multi-field micropolar models. Hypotheses mentioned in Sec. 1.2 are mutually independent, complementary and can be used in various combinations. In Sec. 2.3.1 and 2.3.2 we use ideas of micropolar and multi-field approaches. Keeping derivatives up to the fourth order in expansion (3) leads to the hierarchical system higher-order gradient multi-field micropolar models [8].

2.4. Plane wave solutions. The comparative analysis of the models. We compare models by using plane wave solutions

$$\begin{bmatrix} u_{k,m}(t) \\ v_{k,m}(t) \\ \varphi_{k,m}(t) \end{bmatrix} = \begin{bmatrix} U \\ V \\ \Phi \end{bmatrix} \exp[i(\omega t - kK_x - mK_y)], \qquad \begin{bmatrix} u(x,y,t) \\ v(x,y,t) \\ \varphi(x,y,t) \end{bmatrix} = \begin{bmatrix} U \\ V \\ \Phi \end{bmatrix} \exp[i(\omega t - k_x x - k_y y)]$$

The dispersion curves of the conventional and higher-order gradient single-field models coincide with the dispersion curves of the discrete system at the point (K_x, K_y)=(0, 0) and approximate them around this point [3]. The higher-order gradient model improves the accuracy of the approximation at this point in comparison with the classical micropolar model. However, for short wavelength waves both single-field micropolar models produce results with an essential error.

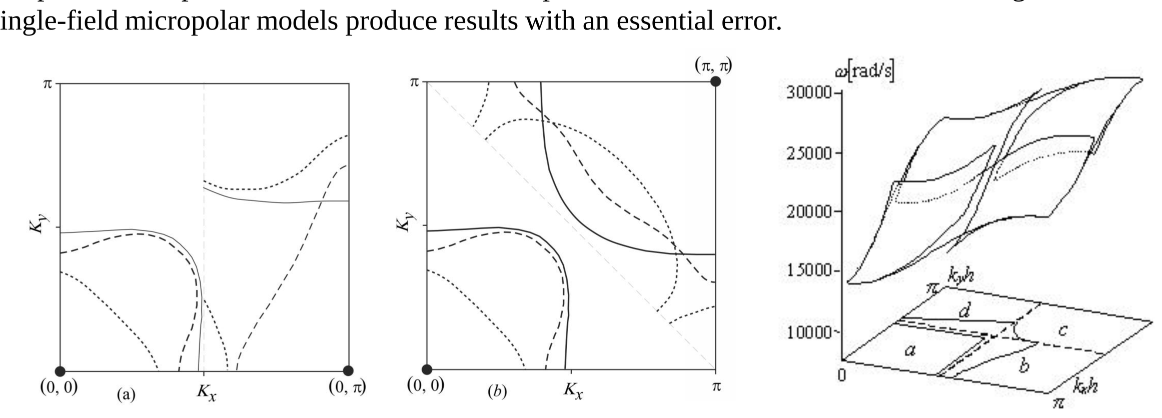


Fig. 5. Area of the first Brillouin zone where relative error $|(\omega_i^{cont.} - \omega_i^{discr.})/\ \omega_i^{discr}|$ for the models correspond to the macro-cells "b" and "c" is smaller than 5%. The dispersion surfaces of the micro-rotational waves of the square lattice and four-field Cosserat model.

The two-field model includes the equations of the single-field model and additional equations. Six dispersion surfaces for the two-field models can be split into two groups. The surfaces of the first group correspond to those of the single-field model. Thus, two-field models possess the properties of single-field models and provide a good approximation of the dispersion surfaces for the discrete system for long wavelength waves. The dispersion relations of the second group of surfaces correspond to additional equations of the two-field model. The comparison shows that the models "b", "c", and "d" specify the single-field micropolar model for short waves with the wave numbers around the corners (π, 0), (π, π), and (0, π) of the first Brillouin zone, respectively [5]. Figures 5a and 5b show area of the first Brillouin zone where relative error $|(\omega_i^{cont.} - \omega_i^{discr.})/\ \omega_i^{discr}|$ for the models correspond to the macro-cells "b" and "c" is smaller than 5%.

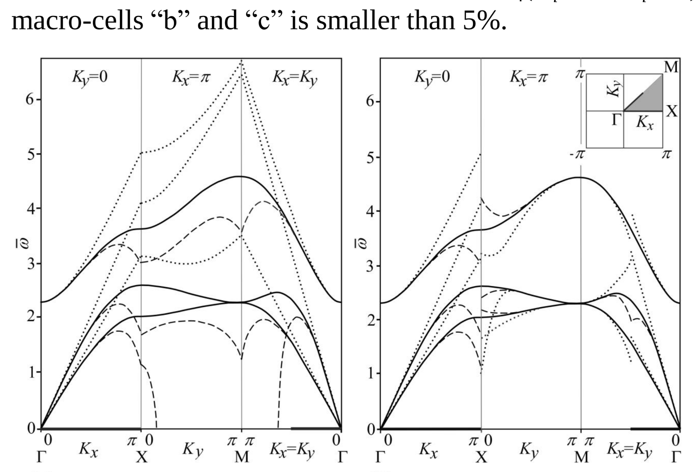


Fig. 6. Figure illustrates the results of the comparative analysis of the models, the accuracy of the approximation of the dispersion surfaces of the discrete system through the dispersion surfaces of (a) single-field and (b) "c" two-field models, and the influence on accuracy of the order of derivatives in the models. The dispersion curves for the discrete system in the sections K_y = 0, K_x = π, K_x = K_y are represented by solid lines. The same curves obtained for the field models with derivatives up to second and fourth orders are represented by dotted and dashed lines, respectively.

2.5. One-dimensional solutions for thin layer (structural interface). Also generalized models are more general and include conventional model, they are more complex and one of important questions in generalized continuum mechanics is the questions of Sec. 1.3.: where the model give new results, can be and should be used? We consider the one-dimensional deformations of a lattice placed between two rigid components (see Fig. 7) [8]. Assuming that the generalized displacements are constant for elements along the diagonals, i.e. for $k+m$ = *const*, we denote components $u_{k,m}$, $v_{k,m}$, and $\varphi_{k,m}$ by using the abbreviated notations U_m, V_m, and Φ_m. The equations for U_m and V_m, Φ_m are decoupled, and we will concentrate on the solutions for U_m only.

In the new co-ordinates $O\xi\eta$, discrete equation of motion has the form

$$M\ddot{U}_m = (K_n + K_s)(U_{m-1} - 2U_m + U_{m+1}) + K_n^d(U_{m-2} - 2U_m + U_{m+2}) \tag{4}$$

Correspondent one-dimensional equations of the single-field higher-order gradient theory have the form

$$MU_{tt} = (K_n + K_s + 4K_n^d)H^2U_{\xi\xi} + \frac{1}{12}(K_n + K_s + 16K_n^d)H^4U_{\xi\xi\xi\xi} \tag{5}$$

This equation can be obtained independently by using Taylor series expansions in Eq (4).

Similarly, changing the variables and considering one-dimensional displacements, additional equation of 2D two-field model (model "c" in Sec. 2.3), leads to the second equation of the two-field model

$$M\tilde{U}_{tt} = -4(K_n + K_s)\tilde{U} - (K_n + K_s - 4K_n^d)H^2\tilde{U}_{\xi\xi} - \frac{1}{12}(K_n + K_s - 16K_n^d)H^4\tilde{U}_{\xi\xi\xi\xi}. \tag{6}$$

In order to explain the notations and to illustrate the method of derivation of the multi-field models described in Sec. 2.3, we will obtain Eqs. (5) and (6) of the two-field model directly from Eq. (4).

Although the unit cell in the problem under consideration consists of the single layer, we assume that a cell of periodicity consists of two layers and use the notations $U_{2n}^{[1]}(t)$ and $U_{2n+1}^{[2]}(t)$ for displacements of the layers with coordinates ξ=2nH and ξ=(2n+1)H, respectively. Eq. (4) can thus be rewritten in the form

$$M\ddot{U}_{2n}^{[1]} = (K_n + K_s)\left(U_{2n-1}^{[2]} - 2U_{2n}^{[1]} + U_{2n+1}^{[2]}\right) + K_n^d\left(U_{2n-2}^{[1]} - 2U_{2n}^{[1]} + U_{2n+2}^{[1]}\right), \tag{7}$$

$$M\ddot{U}_{2n+1}^{[2]} = (K_n + K_s)\left(U_{2n}^{[1]} - 2U_{2n+1}^{[2]} + U_{2n+2}^{[1]}\right) + K_n^d\left(U_{2n-1}^{[2]} - 2U_{2n+1}^{[2]} + U_{2n+3}^{[2]}\right). \tag{8}$$

We use two functions $U^{[1]}(x, t)$ and $U^{[2]}(x, t)$ in order to describe displacements of odd and even layers. The Taylor series expansions of the displacements in Eqs. (7) and (8) up to fourth order terms around the points for which these equations were obtained, gives the system of coupled equations for the two-field model

$$MU_{tt}^{[1]} = \mathrm{L}U^{[1]} - \mathrm{L}_*\left(U^{[1]} - U^{[2]}\right), \qquad MU_{tt}^{[2]} = \mathrm{L}U^{[2]} + \mathrm{L}_*\left(U^{[1]} - U^{[2]}\right), \tag{9}$$

where we separate the operator L for the single-field model, Eq. (5), from the additional operator L*, which describes the interaction of the fields. This representation of the model can be useful for the interpretation and the generalization of two-field model, in particular in the presence of nonlinearities. In the linear case, one can to split the system of coupled equations (9) in two independent equations (5) and (6) by introducing the new field functions

$$U = \frac{1}{2}\left(U^{[2]} + U^{[1]}\right), \qquad \tilde{U} = \frac{1}{2}\left(U^{[2]} - U^{[1]}\right). \tag{10}$$

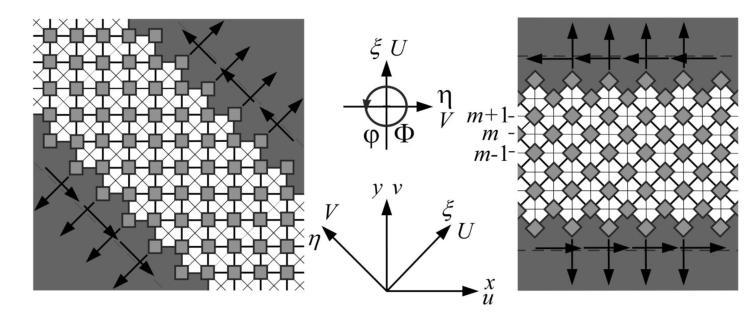


Fig. 7. Representation of a thin lattice layer in a problem of tension between two rigid parts in different coordinate systems.

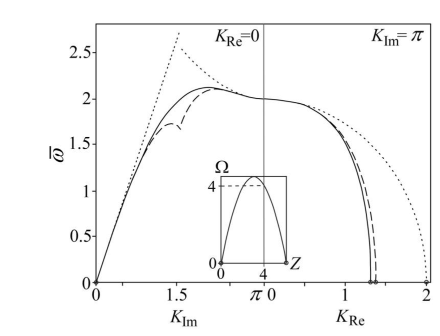


Fig. 8. Dispersion curves of discrete layer system for harmonic, K_{Re}=0, and localized short-wave solutions, K_{Im}=0 (solid lines). The same curves obtained by using two-field models with derivatives up to second and fourth orders (dotted and dashed lines, respectively).

2.5.1. Short-wave dynamical and static solutions. Comparison of the models. The analysis of the discrete, single- and two-field models is based on the solutions of the form $U_m(t)=Ue^{i\omega t-Km}$ and $U(x, t)=Ue^{i\omega t-Kx/H}$ with complex value $K=K_{Re}+iK_{Im}$. The case of harmonic waves, K_{Re}=0, was considered in Sec. 2.5 and illustrated in Fig. 6 on interval MΓ. It is interesting compare models for localized solutions, when K_{Re} is not equal to zero. Analysis show that discrete model have branch of short wave solutions $K_{Im}=\pi$ (solid line in Fig. 8). Single-field models with derivatives up to second and even forth orders do not give such solutions, while two field model give it. Correspondent curves for two-field model with derivatives of the second and fourth orders are shown by dotted and dashed lines in Fig. 8. Comparison for short-wave localised static solutions is presented in Figs. 9 and 10.

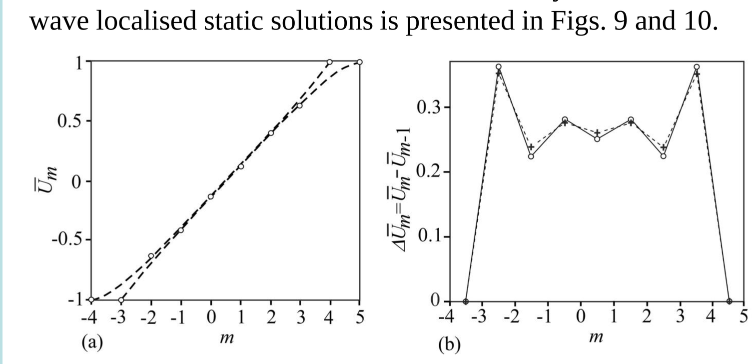


Fig. 9. (a) Displacements of layers of the lattice (circles). Their approximations by using two slowly varying functions in the two-field model (dashed lines). (b) The differences of the displacements of neighboring elements calculated by using the discrete and two-field models (circles and crosses, respectively). Continuous and dashed lines are drawn to underline short wavelength behavior of the solutions near the boundaries.

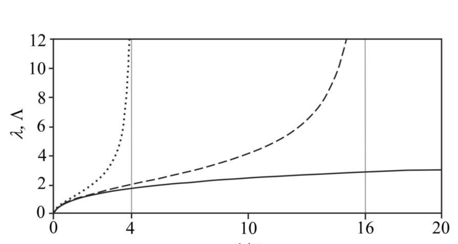
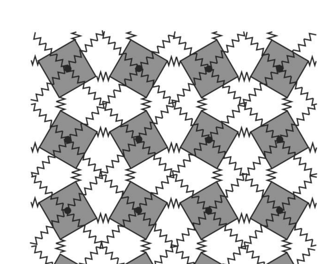


Fig. 10. The dependencies of the localization parameters, λ and Λ, on the parameter $1/\gamma=(K_n+K_s)/K_n^d$ of the discrete system calculated by using the discrete model (solid line) and two-field models with derivatives up to second (dotted line) and fourth (dashed line) orders.

3. Conclusion

The N-field theory is obtained as a continuum analogue for the discrete model with a periodic cell containing N primitive cells by using N vector fields to describe deformation. This approach gives the possibility to construct a hierarchy of models with increasing complexity and accuracy. By increasing the number of fields, the multi-field approach gives a natural way to describe both long- and short wavelength deformations. The latter ones are often considered as inaccessible for continuum models. However, in some cases such deformations may become very important, in particular in fracture, instability, and plasticity problems. Since the multi-field theory is valid for both long and short waves, it is an appropriate theory to describe the coupling between effects on macro- and structural levels.

Two examples of Cosserat solids with unusual properties composed from the finite size particles for which described theory may be useful: auxetic materials, i.e. materials with negative Poisson's ratio (Fig. 11) [6, 7], and materials with chiral microstructure (Fig. 12).



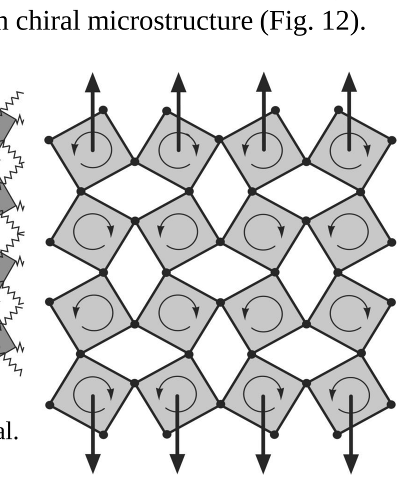


Fig. 11. Auxetic material.

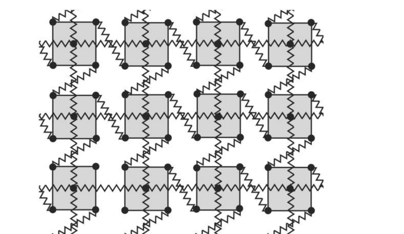


Fig. 12. Material with chiral structure.

[1] Cosserat E., Cosserat F., *Théorie des Corps Déformables*, Hermann A. et Fils, Paris, 1909.
[2] Eringen A.C., *Microcontinuum Field Theories: Foundations and Solids*, Springer-Verlag, New York, 1999.
[3] Suiker A.S.J., Metrikine A.V., de Borst R.: Comparison of wave propagation characteristics of the Cosserat continuum model and corresponding discrete lattice models. *International Journal of Solids and Structures* **38**: 1563-1583, 2001.
[4] Pavlov I.S., Potapov A.I., Maugin G.A.: A 2D granular medium with rotating particles. *International Journal of Solids and Structures* **43**: 6194-6207, 2006.
[5] Vasiliev A.A., Miroshnichenko A.E.: Multi-field modelling of Cosserat solids. *Journal of the Mechanical Behavior of Materials* **16(6)**: 379-392, 2005.
[6] Vasiliev A.A., Dmitriev S.V., Ishibashi Y., Shigenari T.: Elastic properties of a two-dimensional model of crystals containing particles with rotational degrees of freedom. *Physical Review* **B 65**: 094101, 2002.
[7] Vasiliev A.A., Dmitriev S.V., Miroshnichenko A.E.: Multi-field continuum theory for medium with microscopic rotations. *International Journal of Solids and Structures* **42**: 6245-6260, 2005.
[8] Vasiliev A.A., Miroshnichenko A.E., Ruzzene M.: Multi-field model for Cosserat media. *Journal of Mechanics of Materials and Structures* (in press).