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Collapse results for query languages in database theory

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Abstract. This is a survey of collapse results obtained mainly by members of the Tver State University seminar on the theoretical foundations of computer science. Attention is focused on the relative isolation and pseudo-finite homogeneity properties and universes without the independence property. The Baldwin–Benedikt reducibility theorem is proved for these universes. The Dudakov boundedness theorem is proved for reducible theories. The relative isolation theorem is proved for reducible and bounded theories, and as a consequence the collapse theorem is obtained for reducible theories. It is noted that reducibility is equivalent to the relative isolation property. On the other hand, results of Dudakov are presented showing that the effectively reducible theories having an effective almost indiscernible sequence admit an effective collapse of locally generic queries using not only ordering and names of stored tables but also relations and operations of the universe, into queries not using the relations and operations of the universe. Also presented is Dudakov's example of an enrichment of the Presburger arithmetic for which the collapse theorem fails but the elementary theory of the enrichment is decidable. This answers some open questions in the negative.

Contents

1. Introduction

Since the times of Codd, a typical model of database is the relational model in which a database is regarded as a family of finitely many finite tables (see [1], [2]). This model is realized in the majority of existing tools of database control and

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in the proposed query languages. Here it is customary to propose some pastiche of the language of first-order predicate logic for the query language. This tradition also goes back to Codd, who proposed using as a query language the language of relational expressions, which is practically equivalent to the language of first-order predicate logic.

It is usually convenient to assume here that the elements of the tables stored are taken from a fixed set, the so-called universe. For example, the universe can be taken to be the set of positive integers, the set of all words of some finite alphabet, or some other set. This set must be infinite. It can be equipped with its own relations and operations, which form the signature of the universe. As a rule, these relations and operations by their nature cannot be given by finite tables.

Thus, databases are intended for storing current information concerning a data domain structured in some way. At any moment of time the information is *finite* and is given by a *finite* family of *finite* tables ([1], [2]). As a rule, the number of tables and the structure of every table are preserved in the course of time, but the rows of the tables change. New rows can be added, and some old rows can be deleted. The rows of the tables stored are finite sequences of elements. The number of elements in each sequence is fixed for a particular table. Practically, the structure of a table is the number of elements in each row of the table. More formally (or more scientifically), every table is a finitary finite relation, and the database itself is a finite family of finitary finite relations. To make discussions of a database more convenient, one can assign to any relation in the database a name with an indication of the number of arguments (or arity) of the name. The *scheme* (or *signature*) of a database is the finite sequence of these names of relations equipped with an indication of the arity of every name. At each moment of time, some relations of these arities correspond to the names of the relations in this scheme. This defines the *state* of the database at the given instant.

A state is said to be *finite* if all its relations are finite (every table contains finitely many rows). It is sometimes convenient to consider states of the database that are constrained by some conditions rather than arbitrary states. A typical constraint is that the elements of all rows of all tables are chosen in a fixed subset I of the universe. In other words, corresponding to each name of a relation in the database scheme is a relation of the same arity on the set I. In this case we say that the database state is a state over I.

We consider only linearly ordered universes. This term is used for universes for which the family of relations of the universe contains a binary relation that is a linear order relation (a linear, transitive, and antisymmetric relation). We use the symbol $<$ as the name of this linear order.

Thus, queries are formulae of first-order predicate logic.

For linearly ordered universes we consider *locally generic queries* that are preserved under every order-preserving map of a finite subset of the universe into the universe. Roughly speaking, the answer to such a query is based on the stored information but does not depend on the way this information is encoded when stored. For a more precise definition, see § 4.

In a query language, along with the names of tables to be stored, one can also use the names of relations and operations of the universe itself. In other words, in the query language one can consider both stored information and general knowledge about the universe. For instance, the language SQL used in the Oracle system and in similar systems allows one to use the names of stored tables, the comparison relation, and the arithmetic operations on numbers.

Queries in which both the names of relations in the database scheme and the names of relations and operations of the universe itself are used are said to be *extended*. Queries in which only the symbol $<$ and the names of relations in the database scheme are used are said to be *restricted*. The statement that every locally generic extended query is equivalent for finite states of the database to a restricted query is referred to as a *collapse* theorem or a collapse result for the universe under consideration.

Does the use of general knowledge increase the expressive power of a query language?

One usually considers ordered universes. A more specific question is as follows: Does the use of the order relation in the universe extend the expressive power of a query language?

It is clear that one must consider only queries independent of the ordering in the universe, in other words, invariant under any permutation of the elements of the universe. These queries are said to be $=$-generic.

The answer was obtained by Yuri Gurevich (see [3]). This answer does not depend on the universe and is positive. Even if we know nothing about the order, we can always propose an $=$-generic query that cannot be written without using the order relation (see [4]).

Surprisingly, the following collapse theorem holds in many cases: the use of other general knowledge, in addition to the ordering, does not extend the expressive power of a query language. For details, see [3]–[12].

This is not the case for other theories, for instance, for elementary arithmetic or the theory of hereditary finite sets; the use of knowledge of these universes increases the abilities of the first-order language. There are also decidable examples of these theories (see [13]). In some cases (see [14]), increasing the expressive power depends on the signature of a database.

The present paper is devoted to the problem of describing the universes for which the collapse theorem holds.

A criterion for a query to be equivalent to a restricted query was found in [3], where the pseudo-finite homogeneity property and the isolation property were also introduced (the validity of one of these properties in a universe ensures the validity of the collapse theorem for this universe). On the other hand, in [5] the collapse theorem was proved for universes without the independence property. How are these conditions related?

In [15] the somewhat more restrictive notions of (M, I)-pseudo-finite homogeneity and (M, I)-isolation were proposed; in these notions, one considers states over a chosen indiscernible sequence rather than over all indiscernible sequences. In many cases this helps to significantly simplify the proof of the collapse theorem (see, for instance, [10], [7]). We present the results of [15] in a simplified and more accurate form.

In particular, it turns out that the universes without the independence property satisfy the isolation condition, and thus the validity of the collapse theorem for these universes is a corollary to our theorems.

The formulae of first-order predicate logic of signature consisting solely of $<$ are called order formulae. The formulae of first-order predicate logic of signature L are called L-formulae. Let (M, I) be an enrichment of a system M of signature L to a system of signature (L, P) formed by adding the subset I as an interpretation for a unary relation P. One says that the system (M, I) is *reducible* if

for any L-formula $\phi(\overline{x}, \overline{y})$ there is an order formula $\psi(\overline{w}, \overline{y})$ such that for any sequence $\overline{m}$ of elements in M there is a sequence $\overline{c}_{\overline{m}} \in I$ such that

$$(\forall\, \overline{y} \in P)(\psi(\overline{c}_{\overline{m}}, \overline{y}) \leftrightarrow \phi(\overline{m}, \overline{y})).$$

A universe U is said to be *reducible* if there is a reducible system (M, I) such that U and M are elementarily equivalent and I is an infinite indiscernible sequence in M.

An (L, P)-formula is said to be *P-bounded* if it does not contain P or is of the form

$$(\forall\, x \in P)\Psi \quad \text{or} \quad (\exists\, x \in P)\Psi,$$

where Ψ is a P-bounded formula. A system (M, I) of signature (L, P) is said to be *bounded* if every (L, P)-formula is equivalent in (M, I) to some P-bounded formula.

The proof in [5] of the collapse theorem for universes without the independence property consists of two parts. One first proves that such a universe is reducible. Then one proves that these universes are bounded, and the proof uses the absence of the independence property again.

As was proved already in [15],[1] reducible and bounded universes possess the collapse theorem. Here we prove the Dudakov theorem that every reducible universe is bounded. Thus, the collapse theorem holds for all reducible universes. This leads to a simpler proof of the collapse theorem for universes without the independence property. As a by-product, we note that relative isolation is equivalent to reducibility.

By the *active* domain of a state of the database we mean the family of all elements of the universe that are in at least one row of at least one table. A query is said to be *active* if all its quantifiers are restricted to the active domain. We prove the Dudakov theorem that in every reducible universe each extended query locally generic for finite states is equivalent for finite states to some active query. For reducible universes in which one can effectively construct sequences indiscernible for a given formula, this provides an effective collapse of extended queries into restricted ones, an achievement impossible in principle using the old methods.

Many examples of enrichments of the Presburger arithmetic by a single unary operation with decidable elementary theory were indicated in [10]. As was noted there, the collapse theorem holds for each of these examples. A fundamentally different example of enriching the Presburger arithmetic by a single unary operation with decidable elementary theory was proposed in [16]. We suggest an improvement of this example and present the Dudakov theorem that the collapse theorem fails to hold for this example. This disproves the well-known conjecture (see [5], [4], [10]) that all enrichments of the Presburger arithmetic with decidable elementary theory satisfy the collapse theorem.

[1] *Russian Editor's note*: In other terms.

2. Preliminaries from set theory

We assume that the reader is acquainted with the preliminaries of naive set theory.

We identify any ordinal with the set of all smaller ordinals and any cardinal with the least ordinal of the corresponding cardinality. As usual, $\aleph$ stands for an infinite cardinal. Infinite cardinals can be indexed by ordinals. In particular, $\aleph_0$ stands for the least infinite cardinal, $\aleph_{\alpha+1}$ is the least cardinal exceeding $\aleph_\alpha$, and for any limit ordinal α the symbol $\aleph_\alpha$ denotes the least cardinal exceeding all cardinals of the form $\aleph_\beta$ with $\beta < \alpha$.

At the same time, the least infinite cardinal is also the least infinite ordinal and is also denoted by ω. All finite ordinals are cardinals and are called natural numbers. Thus, the symbols ω and $\aleph_0$ denote the set of natural numbers.

If the cardinality of a set A is $\aleph$, then we denote the cardinality of the set of all subsets of the set A by $2^\aleph$.

For a cardinal κ we denote by κ^+ the least cardinal exceeding κ. By $\aleph^*_\alpha$ we denote the sum $\sum_{\beta<\alpha} 2^{\aleph_\beta}$. We set $\beth_0 = \aleph_0$ and $\beth_{\xi+1} = 2^{\beth_\xi}$. For any limit cardinal ξ we set

$$\beth_\xi = \bigcup_{\zeta<\xi} \beth_\zeta.$$

Let α be a limit ordinal. The *confinality* of α is the ordinal $\mathrm{cf}(\alpha)$ equal to the least ordinal β for which there is a function f from β to α such that

$$\bigcup_{\zeta<\beta} f(\zeta) = \alpha.$$

A cardinal κ is said to be *regular* if $\kappa = \mathrm{cf}(\kappa)$.

Since $\mathrm{cf}(\alpha) = \mathrm{cf}(\mathrm{cf}(\alpha))$, it follows that $\mathrm{cf}(\alpha)$ is a regular cardinal for any ordinal α.

The following four statements have simple proofs. The proofs of the last two statements are presented, for example, in the book [17]. The proof of the first statement was presented in the book [18]. We present these proofs here only for completeness of our presentation. The second statement is proved here.

Theorem 2.1. $\mathrm{cf}(\beth_\alpha) = \mathrm{cf}(\alpha)$ *for any limit ordinal* $\alpha > 0$.

Proof. If there is a function f from β to α such that

$$\bigcup_{\zeta<\beta} f(\zeta) = \alpha,$$

then, setting $g(\gamma) = \beth_{f(\gamma)}$, we see that

$$\bigcup_{\zeta<\beta} g(\zeta) = \beth_\alpha.$$

Conversely, if there is a function g from β to $\beth_\alpha$ such that

$$\bigcup_{\zeta<\beta} g(\zeta) = \beth_\alpha,$$

then, taking $f(\gamma)$ to be the greatest ζ such that $g(\gamma)$ contains $\beth_\zeta$, we see that

$$\bigcup_{\zeta<\beta} f(\zeta) = \alpha.$$

Theorem 2.2. *$\beth_\alpha^* = \beth_\alpha$ for any limit ordinal α.*

Proof. If $\lambda < \beth_\alpha$, then $\lambda < \beth_\beta$ for some $\beta < \alpha$. Therefore, $2^\lambda \leqslant 2^{\beth_\beta} = \beth_{\beta+1}$. This shows that $\beth_\alpha^* \leqslant \beth_\alpha$. The converse inequality holds for all cardinals.

Theorem 2.3 (König). *Let κ_i and λ_i be cardinals for all $i \in I$. If $\kappa_i < \lambda_i$ for all $i \in I$, then*

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i.$$

Proof. For any $i \in I$ we choose a set C_i of cardinality κ_i and a set T_i of cardinality λ_i. We assume that C_i is a subset of T_i and C_i and C_j are disjoint for distinct i and j. An arbitrary element x in the union of the sets C_i for $i \in I$ belongs to one of these sets, say C_j. To this element x we assign the element t_x of the Cartesian product $\prod_{i\in I} T_i$ such that $t_x(j) = x$ and $t_x(i) \in (T_i \setminus C_i)$ for $i \in (I \setminus \{j\})$. It is clear that t_x and t_y are distinct for distinct x and y. This proves that

$$\sum_{i\in I} \kappa_i \leqslant \prod_{i\in I} \lambda_i.$$

We claim now that the set $\prod_{i\in I} T_i$ cannot be represented as a union of disjoint sets Z_i of cardinality κ_i for $i \in I$. Suppose the contrary. Let a representation of this kind be possible, and let

$$U_i = \{t(i) \mid t \in Z_i\}.$$

It is clear that the cardinality of U_i does not exceed κ_i. We take an element $f \in \prod_{i\in I} T_i$ such that $f(i) \in (T_i \setminus U_i)$. The element f thus constructed does not belong to any Z_i. This contradiction proves the theorem.

Corollary 2.4. $\mathrm{cf}(2^{\aleph_\alpha}) > \aleph_\alpha$.

Proof. We must prove that

$$\sum_{\beta<\aleph_\alpha} \kappa_\beta < 2^{\aleph_\alpha}$$

if each κ_β is less than $2^{\aleph_\alpha}$. However, by König's theorem we have

$$\sum_{\beta<\aleph_\alpha} \kappa_\beta < \prod_{\beta<\aleph_\alpha} 2^{\aleph_\alpha} = (2^{\aleph_\alpha})^{\aleph_\alpha} = 2^{\aleph_\alpha}.$$

Theorem 2.5. *There are cardinals $\kappa = \kappa^*$ with arbitrarily large confinality.*

Proof. It follows from the previous four statements that, setting $\kappa = \beth_{2^{\aleph_\alpha}}$, we have $\kappa^* = \kappa$ and $\mathrm{cf}(\kappa) > \aleph_\alpha$. This just means that there are cardinals $\kappa = \kappa^*$ with arbitrarily large confinality.

We need the following theorem.

Theorem 2.6 (Ramsey). *If the set of all n-element subsets of an infinite set A is partitioned into k parts, then the set A contains an infinite subset B whose n-element subsets all belong to one of these parts.*

Proof. It suffices to prove the theorem for $k = 2$. We prove the statement by induction on n. The theorem is trivial for $n = 1$. Suppose that it holds for n. Let us prove it for $n + 1$.

It suffices to prove the theorem for a countable set A. We index the elements of A by the natural numbers. Consider the $(n + 1)$-element subsets of A containing the first element a_1 of A and delete the first element from these sets. The n-element subsets thus obtained are divided into two parts. By induction, there is an infinite subset B_2 of $A\backslash\{a_1\}$ whose n-element subsets all belong to one of the parts. Let the elements $a_1, \ldots, a_i$ and an infinite subset B_{i+1} containing none of them be already constructed and let a_{i+1} be the first element in B_{i+1}. Consider the $(n+1)$-element subsets of B_{i+1} that contain a_{i+1} and delete this element from each of them. The n-element subsets thus obtained are divided into two parts. By induction, there is an infinite subset B_{i+2} of $B_{i+1} \setminus \{a_{i+1}\}$ whose n-element subsets all belong to one of these two parts. The sequence of pairwise distinct elements $a_1, \ldots, a_i, \ldots$ thus obtained has the following property: all $(n+1)$-element subsets that consist of elements of the sequence, contain a_i, and do not contain $a_1, \ldots, a_{i-1}$ belong to the same part $k(i)$. It remains to find a subsequence $a_{j_1}, \ldots, a_{j_i}, \ldots$ of the sequence $a_1, \ldots, a_i, \ldots$ such that

$$k(j_1) = \cdots = k(j_i) = \cdots .$$

This proves Theorem 2.6.

We also need the following theorem.

Theorem 2.7 (Ramsey theorem on finite sets). *For any natural numbers n, k, m there is a number $r(n, k, m)$ such that, for any partition of all n-element subsets of a finite set A containing not less than*

$$r(n, k, m)$$

elements into k parts, the set A contains a subset B containing not less than m elements and such that all n-element subsets of B belong to one of these k parts.

Proof. It suffices to prove the theorem for $k = 2$. For $r(1, 2, m)$ one can take $2m$. Suppose that the number $r(n, 2, m)$ has been determined. We are going to compute $r(n+1, 2, m)$. Let $s(n, m, 0) = 2r(n, 2, m)$ and $s(n, m, i+1) = r(n, 2, s(n, m, i))+1$.

Repeating the arguments in the proof of the above theorem of Ramsey, one can easily see that for the number $r(n + 1, 2, m)$ one can take $s(n, m, m + 1)$.

3. Preliminaries from model theory

In this section we recall some preliminaries from model theory. Even if we were relying on the reader to be familiar with the basics of this theory, we would still have to make conventions about the notation. However, we also keep in mind the readers interested in theoretical programming who did not necessarily go through a detailed course in mathematical logic.

Of course, we must first of all fix a data domain whose properties we are going to study. This domain is a non-empty set together with operations and relations defined on it. These operations and relations are said to be *basic*.

To be able to write out the properties under consideration, we need some notation or, in other words, some names for the given basic operations and relations. For instance, the data domain in arithmetic is the set ω of natural numbers, the basic operations are the argument-free operations distinguishing 0 and 1 and also the operations of addition and multiplication, and the basic relation is the order relation. In elementary geometry no operations are considered as a rule, and the basic relations are the identity relation, the incidence relation, the relation of being a point, a line, or a plane, and others.

One can see already from these examples that the number of arguments (that is, the so-called arity) in the operations and relations can be different. For example, the operations of addition and multiplication are binary (have two arguments), and the relation of being a point has one argument.

An *operation* with n arguments on a set A is a map. To any sequence of length n formed by elements of A this map assigns a uniquely defined element of A. For instance, the operation of addition for natural numbers has two arguments, and to any pair of natural numbers it assigns the sum of these numbers. The sum is uniquely defined for any pair of natural numbers, but different pairs can have different sums.

We denote by A^n the set of all sequences of length n formed by elements of the set A.

Every subset of A^n is called a *relation* or a *predicate* of n arguments on the set A. If a sequence $a_1, \dots, a_n$ (of length n) formed by elements of the set A is in the relation, then we say that the relation, call it P, holds true on the sequence, that the relation is equal to 1 on the sequence, or that $P(a_1, \dots, a_n)$ is *true*. If a sequence $a_1, \dots, a_n$ (of length n) formed by elements of A is not in the relation P, then we say that P fails to hold on the sequence, that the relation is equal to 0 on the sequence, or that $P(a_1, \dots, a_n)$ is *false*.

Definition 1. By a *signature* we mean a family of operation and relation symbols together with a map which assigns a natural number to any symbol; the value is the number of argument places, or the arity of the symbol. The argument-free operation symbols are referred to as signature *constants* or symbols of distinguished elements. The cardinality of the family is called the *cardinality of the signature*. Signature symbols are sometimes referred to as *names* of the operations and relations, respectively.

Definition 2. By an *algebraic system* (one sometimes speaks also of a *structure* or *interpretation*) of a signature L one means a non-empty set together with a map which assigns to any relation symbol in L a relation of the same arity on the set, and to any operation symbol in L an operation of the same arity on the same set. This fixed non-empty set is called the *basic set* or *support* of the algebraic system. The cardinality of the support is called the *cardinality of the algebraic system*.

The relation in an algebraic system A corresponding to a relation symbol P in L is denoted by P^A. The operation corresponding in an algebraic system A to an operation symbol f in L is denoted by f^A.

Let us present some examples.

Example 1. Let the basic set be the union of the following subsets:
1) last names of lecturers {Ivanov, Petrov, Sidorov, Stepanov};
2) titles of subjects {algebra, logic};
3) numbers of lecture rooms {201, 202, 203, 204};
4) dates in January {3, 4, 5, 6, 7, 8, 9, 10, 11};
5) identifiers of student groups $\{M1, M2, M3, M4\}$.

Let a signature L_1 consist of symbols of a quaternary relation R and a ternary relation P.

We consider the algebraic system A_1 of signature L_1 with the above basic set in which the relations P and R are interpreted as follows. For every group and every subject the relation P indicates the examiner in this group on this subject and consists of the triples

$$\begin{array}{ll}\langle M1, \text{algebra, Ivanov}\rangle, & \langle M1, \text{logic, Sidorov}\rangle,\\ \langle M2, \text{algebra, Ivanov}\rangle, & \langle M2, \text{logic, Sidorov}\rangle,\\ \langle M3, \text{algebra , Petrov}\rangle, & \langle M3, \text{logic, Stepanov}\rangle,\\ \langle M4, \text{algebra, Petrov}\rangle, & \langle M4, \text{logic, Stepanov}\rangle.\end{array}$$

For every group and every subject the relation R indicates the date and the lecture room of the exam on this subject in this group and consists of the quadruples

$$\begin{array}{ll}\langle M1, \text{algebra}, 3, 201\rangle, & \langle M1, \text{logic}, 9, 201\rangle,\\ \langle M2, \text{algebra}, 9, 202\rangle, & \langle M2, \text{logic}, 3, 202\rangle,\\ \langle M3, \text{algebra}, 5, 203\rangle, & \langle M3, \text{logic}, 11, 204\rangle,\\ \langle M4, \text{algebra}, 9, 203\rangle, & \langle M4, \text{logic}, 3, 203\rangle.\end{array}$$

The algebraic system A_1 describes the schedule of exams in the groups under consideration.

Example 2. The schedule of exams in the same groups can be given in another way by considering the signature L_2 consisting of a symbol Q of a 5-ary relation and an algebraic system A_2 of signature L_2 such that the basic set of the system A_2 coincides with the basic set of the system A_1 and the relation Q in A_2 consists of the quintuples

$$\begin{array}{ll}\langle M1, \text{algebra, Ivanov}, 3, 201\rangle, & \langle M1, \text{logic, Sidorov}, 9, 201\rangle,\\ \langle M2, \text{algebra, Ivanov}, 9, 202\rangle, & \langle M2, \text{logic, Sidorov}, 3, 202\rangle,\\ \langle M3, \text{algebra, Petrov}, 5, 203\rangle, & \langle M3, \text{logic, Stepanov}, 11, 204\rangle,\\ \langle M4, \text{algebra, Petrov}, 9, 203\rangle, & \langle M4, \text{logic, Stepanov}, 3, 203\rangle.\end{array}$$

Each quintuple contains the identifier of a group, title of a subject, last name of a lecturer, date of an exam, and a lecture room.

A subset of the support of an algebraic system is said to be *closed with respect to an operation* defined on the basic set of the system if the value of the operation belongs to the subset when the values of the arguments belong to the subset. In particular, for any constant this means that the subset under consideration contains the value of this constant. By the *restriction* of an operation f with m arguments to a set D closed with respect to f we mean an operation g defined on D, where g has the same number of arguments as f and satisfies the condition $g(a_1, \dots, a_m) = f(a_1, \dots, a_m)$ for any $a_1, \dots, a_m$ in D. By the *restriction* of an m-ary relation P to D we mean the intersection $P \cap D^m$.

The operations corresponding to the signature symbols of operations in the algebraic system under consideration are called the *basic operations* of this algebraic system. Here the elements corresponding to the signature constants are also called the *distinguished elements* of this algebraic system.

If a subset of the support of an algebraic system A of signature L is closed with respect to all basic operations of this algebraic system, then this subset, together with the restrictions to it of all the basic operations and relations, forms an algebraic system of signature L which is called a *subsystem* of the system A. The notation $A \subseteq B$ means that A is a subsystem of a system B. If A is a subsystem of a system B, then B is said to be an *extension* of the system A.

Let us now pass to the definition of first-order predicate logic.

The symbols for superpositions of basic operations are usually referred to as *terms*.

In terms we can use *object variables*, that is, variables taking values in the basic set of the algebraic system under consideration. We shall denote the object variables by lower-case Latin letters $x, y, z, \dots$, with subscripts.

Definition 3. a) Every object variable and every 0-ary operation symbol of signature L are terms of signature L. Any object variable appears in itself, and other variables do not appear in the object variable. No object variable appears in any operation symbol.

b) If f is an n-ary operation symbol of signature L and if $t_1, \dots, t_n$ are terms of signature L, then $f(t_1, \dots, t_n)$ is also a term of signature L. A variable occurs in $f(t_1, \dots, t_n)$ if and only if this variable occurs in at least one of the terms $t_1, \dots, t_n$.

c) Each expression is a term of signature L only if this can be proved by using items a) and b).

A term containing no variables is said to be *closed*.

As a rule, if f is binary, then one writes $(t_1 \, f \, t_2)$ instead of $f(t_1, t_2)$. For instance, one writes $(x_1 + x_2)$ instead of $+(x_1, x_2)$.

In particular, $((x_1 + x_2) \times x_3)$ is an arithmetic term in which $+$ and $\times$ are the symbols of the binary operations of addition and multiplication.

We note that if a signature contains no operation symbols, then every term is simply an object variable, because in this case one cannot use the item b) of the definition.

Definition 4. By a *memory state* (or evaluation) of an algebraic system we mean a map assigning to each object variable some element of the basic set of the system.

Definition 5. Let σ be a state of an algebraic system A of signature L.

a) If x is an object variable, then $\sigma(x)$ is given by the definition of σ.

b) If f is a 0-ary operation symbol of signature L and f^A is an argument-free operation assigned to the symbol f in the algebraic system A, then $\sigma(f)$ is the value of f^A.

c) If f is an n-ary ($n > 0$) operation symbol of signature L, f^A is the operation corresponding to the symbol f in the algebraic system A, and $t_1, \dots, t_n$ are terms of signature L, then

$$\sigma(f(t_1, \dots, t_n))$$

is

$$f^A(\sigma(t_1), \dots, \sigma(t_n)).$$

The element $\sigma(t)$ is called the *value of the term* t at the state σ.

For example, if $\sigma(x_1) = 1$, $\sigma(x_2) = 2$, and $\sigma(x_3) = 7$, then $\sigma(((x_1+x_2)\times x_3)) = 21$ under the standard interpretation of the symbols $+$ and $\times$.

To represent properties of elements of an algebraic system, one can use formulae.

Definition 6. a) If t_1 and t_2 are terms of signature L, then $t_1 = t_2$ is called an *atomic formula* of signature L. A variable appears in the formula $t_1 = t_2$ if and only if it appears in either t_1 or t_2.

b) If P is a 0-ary relation symbol of signature L, then P is an atomic formula of signature L, and no object variable appears in this formula.

c) If P is an n-ary ($n > 0$) relation symbol of signature L and $t_1, \dots, t_n$ are terms of signature L, then

$$P(t_1, \dots, t_n)$$

is an atomic formula of signature L. A variable appears in this formula if and only if it appears in at least one of the terms $t_1, \dots, t_n$.

For a binary symbol P one often writes $t_1 P t_2$ or $(t_1 P t_2)$ instead of $P(t_1, t_2)$, for instance, one writes $t_1 < t_2$ instead of $< (t_1, t_2)$.

Definition 7 (formulae of predicate logic). a) An atomic formula of signature L is a formula of signature L. Any variable appearing in an atomic formula has free occurrences and has no bound occurrences.

b) If Φ and Ψ are formulae of signature L, then $\neg\Phi$, $(\Phi \vee \Psi)$, $(\Phi \wedge \Psi)$, and $(\Phi \to \Psi)$ are also formulae of signature L. An occurrence of a variable in $\neg\Phi$ is free (bound) if and only if the occurrence of this variable in Φ is free (bound, respectively). An occurrence of a variable in $(\Phi \vee \Psi)$, $(\Phi \wedge \Psi)$, $(\Phi \to \Psi)$ is free (bound) if and only if the occurrence of this variable in at least one formula Φ or Ψ or in both the formulae is free (bound, respectively).

c) If Φ is a formula of signature L and x is a variable, then $(\forall\, x)\Phi$ and $(\exists\, x)\Phi$ are also formulae of signature L. The occurrence of the variable x in each of these formulae is bound, and there is no free occurrence of the variable in the formulae. An occurrence of any variable distinct from x in $(\forall\, x)\Phi$ and $(\exists\, x)\Phi$ is free (bound) if and only if the occurrence of this variable in Φ is free (bound).

d) Each expression is a formula of signature L only if this can be proved by using a), b), and c).

Any formula without free occurrences of variables is said to be *closed*.

The symbols $\wedge$, $\vee$, $\to$, and $\neg$ are usually called (propositional) *connectives* and the symbols $\forall$ and $\exists$ are called *symbols of quantifiers*. Here $\forall$ is the symbol of the *universal quantifier* and $\exists$ is the symbol of the *existential quantifier*. If x is a variable, then the expressions $(\forall x)$ and $(\exists x)$ are universal and existential quantifiers, respectively, with respect to the variable x.

The formulae $(\forall x)\Phi$ and $(\exists x)\Phi$ are obtained from Φ by quantification.

The formula Φ in the formulae $(\forall x)\Phi$ and $(\exists x)\Phi$ is called the *domain* of the quantifier $(\forall x)$ or $(\exists x)$, respectively. Roughly speaking, a bound occurrence of a variable is an occurrence in the domain of a quantifier with respect to the variable.

Let us present some examples.

Example 3. Consider the signature L_1 described in Example 1. The expression

$$(\forall z)(\forall x)(\forall y)(((\exists x_1)(\exists y_1)R(x,z,x_1,y_1) \wedge (\exists z)(\exists x_2)(\exists y_2)R(y,z,x_2,y_2)) \\ \to (\exists x_2)(\exists y_2)R(y,z,x_2,y_2)) \tag{1}$$

is a formula of signature L_1 with bound occurrences of the variables x, y, z, x_1, y_1, x_2, y_2 and without free variables.

Example 4. Consider the signature L_2 described in Example 2. The expression

$$((\exists x_1)(\exists y_1)(\exists z_1)Q(x_3,x_1,y,y_1,z_1) \wedge (\exists x_2)(\exists y_2)(\exists z_2)Q(x_1,x_2,y,y_2,z_2))$$

is a formula of signature L_2 with bound occurrences of the variables x_1, y_1, z_1, x_2, y_2, and z_2 and with free occurrences of the variables x_3, y, and x_1. This example shows that the same variable can have both free and bound occurrences in a formula.

Definition 8 (values of formulae). Let A be an algebraic system of a signature L, let Φ be a formula of signature L, and let σ be a state of the system A.

a) If Φ is $t_1 = t_2$, where t_1 and t_2 are terms of signature L, then we have $\sigma(\Phi) = 1$ provided that

$$\sigma(t_1) = \sigma(t_2)$$

and $\sigma(\Phi) = 0$ provided that

$$\sigma(t_1) \neq \sigma(t_2).$$

b) Let Φ be P, where P stands for a 0-ary relation symbol in the signature L, and let P^A be the relation assigned to the symbol P in the system A. Then $\sigma(P)$ is P^A.

c) Let Φ be $P(t_1,\dots,t_n)$, where P is an n-ary relation symbol in the signature L, let $t_1,\dots,t_n$ be terms of signature L, and let P^A be the relation assigned to the symbol P in the system A. In this case we have $\sigma(P(t_1,\dots,t_n)) = 1$ if

$$\langle \sigma(t_1),\dots,\sigma(t_n)\rangle \in P^A$$

and $\sigma(P(t_1,\dots,t_n)) = 0$ if

$$\langle \sigma(t_1),\dots,\sigma(t_n)\rangle \notin P^A.$$

d) If Φ is one of the formulae $\neg\Phi_1$, $(\Phi_1 \vee \Phi_2)$, $(\Phi_1 \wedge \Phi_2)$, $(\Phi_1 \to \Phi_2)$, then $\sigma(\Phi)$ is determined by $\sigma(\Phi_1)$ and $\sigma(\Phi_2)$ according to the rules indicated in Tables 1 and 2.

Table 1

$\sigma(\Phi_1)$	$\sigma(\neg\Phi_1)$
0	1
1	0

Table 2

$\sigma(\Phi_1)$	$\sigma(\Phi_2)$	$\sigma(\Phi_1 \wedge \Phi_2)$	$\sigma(\Phi_1 \vee \Phi_2)$	$\sigma(\Phi_1 \to \Phi_2)$
0	0	0	0	1
0	1	0	1	1
1	0	0	1	0
1	1	1	1	1

e) If Φ is $(\exists\, x)\Phi_1$, then $\sigma(\Phi) = 1$ if and only if there is a state σ_1 of the system A such that $\sigma_1(\Phi_1) = 1$ and $\sigma_1(y) = \sigma(y)$ for any variable y distinct from x.

f) If Φ is $(\forall\, x)\Phi_1$, then $\sigma(\Phi) = 1$ if and only if $\sigma_1(\Phi_1) = 1$ for any state σ_1 of A such that $\sigma_1(y) = \sigma(y)$ for any variable y distinct from x.

If $\sigma(\Phi) = 1$, then we say that Φ is *true* on σ and write $\sigma \models \Phi$. If $\sigma(\Phi) = 0$, then we say that Φ is *false* on σ. If Φ is true on every state of the system A, then we say that Φ is true on A and also that A is a *model* of Φ (or for Φ) and write $A \models \Phi$.

If Φ is true on every algebraic system of signature L, then we say that Φ is *valid*.

It is easy to see that

$$\sigma((\forall\, x)\Phi) = \sigma(\neg(\exists\, x)\neg\Phi),$$
$$\sigma((\exists\, x)\Phi) = \sigma(\neg(\forall\, x)\neg\Phi)$$

for any formula Φ and every state σ.

Formulae Φ and Ψ are said to be *equivalent in an algebraic system* A if $\sigma(\Phi) = \sigma(\Psi)$ for any state σ in A. Formulae Φ and Ψ are said to be *equivalent* if they are equivalent in every algebraic system of the signature under consideration. The notation $\Phi \equiv \Psi$ means that Φ and Ψ are equivalent.

A formula containing no quantifiers is said to be *quantifier-free*. A formula is said to be a *prenex* formula if it is either quantifier-free or of the form $(\forall\, x)\Phi$ or $(\exists\, x)\Phi$, where Φ is a prenex formula and x is a variable. Thus, the first symbols in a prenex formula are quantifiers after which we have a quantifier-free formula. In other words, a prenex formula is obtained from a quantifier-free formula by quantification. In a prenex formula the beginning part formed by quantifiers and standing before a quantifier-free formula is called a *quantifier prefix*.

One can easily prove the following two theorems.

Theorem 3.1. *Every formula is equivalent to some prenex formula.*

Theorem 3.2. *Let all free variables of a formula Φ of signature L be among the variables $x_1, \ldots, x_m$. Let σ_1 and σ_2 be states of an algebraic system A of signature L. If*

$$\sigma_1(x_i) = \sigma_2(x_i)$$

for $i = 1, \ldots, m$, then $\sigma_1(\Phi) = \sigma_2(\Phi)$.

It follows from Theorem 3.2 that the validity of a closed formula depends on the algebraic system itself rather than on a state of the algebraic system.

For example, consider the formula (1) and the algebraic system A_1 described in Example 1. Both the formula and the system A_1 are of signature L_1 described in the same example. The formula (1) is of the form $(\forall z)(\forall x)(\forall y)\Phi$. The formula (1) is true in A_1.

Formulae of signature L are called *L-formulae*. An algebraic system of signature L is also called an *L-system* or an *L-structure*. For an L-formula Φ and a tuple of variables $\overline{x}$ the symbol $\Phi(\overline{x})$ means that the formula Φ contains no free variables not in the tuple $\overline{x}$. If $\overline{a}$ is a tuple of elements of some L-structure and if the length of $\overline{a}$ is equal to that of $\overline{x}$, then, for brevity, we denote by $\Phi(\overline{a})$ the value of $\Phi(\overline{x})$ at any state of this L-structure which assigns the values in $\overline{a}$ to the corresponding variables in $\overline{x}$.

By an *L-theory* one means an arbitrary set of closed L-formulae. An L-theory is said to be *consistent* if there is an algebraic system of signature L on which all formulae of the L-theory are true. This algebraic system is called a *model* of the L-theory. An L-theory is said to be *finitely consistent* if every finite subset of the theory is consistent.

By an *elementary theory* or an *L-theory* of a class K (this theory is denoted by $\mathrm{Th}(K)$) of L-structures we mean the family of all closed L-formulae that are true in all systems of class K. If the class consists of a single L-structure A, then the elementary theory of the class is called the elementary theory (or L-theory) of the L-structure A and is denoted by $\mathrm{Th}(A)$.

Two L-structures are said to be *elementarily equivalent* if every closed L-formula that is true in one of these algebraic systems is true in the other as well. In this case the elementary theories of these L-structures coincide. An L-theory is said to be *complete* if any two models of the theory are elementarily equivalent. We write $A \equiv B$ to mean that the L-structures A and B are elementarily equivalent.

Let a signature L' be a part of the signature L. This means that every operation symbol in L' appears in L as an operation symbol and has in L the same arity (the same number of argument places) as in L', and every relation symbol in L' appears in L as a relation symbol and has in L the same arity as in L'. In this case the signature L' is said to be a *restriction* of L, and L is called an *enrichment* of L'.

Let L' be a restriction of a signature L. In this case starting from any L-structure A, one can obtain an L'-structure by removing the superfluous operations and relations. The L'-structure B thus obtained is denoted by $A \restriction L'$ and is called the *L'-restriction* of the L-structure A. Here the structure A is called an *L-enrichment* of the L'-structure B.

For an L-structure M and an arbitrary subset A of the support of M we denote by $L(A)$ the signature obtained by adding to L the names of all elements of A. These names are symbols of distinguished elements. As a rule, we do not distinguish between an element and its name. If A coincides with the support of an L-structure M, then we write $L(M)$ instead of $L(A)$. Denote by $(M, a \mid a \in A)$ the enrichment of the L-structure M to an $L(A)$-structure in which the value of the added name of an element is the element itself.

For brevity, we sometimes speak of the validity of some closed $L(A)$-formula in the L-structure M, which means the validity of this formula in $(M, a \mid a \in A)$.

An L-structure M is said to be an *elementary subsystem* of an L-structure N, and the L-structure N is said to be an *elementary extension* of the L-structure M, if M is a subsystem of N and every closed $L(M)$-formula ϕ is true in $(M, a \mid a \in |M|)$ if and only if it is true in $(N, a \mid a \in |M|)$. Here the symbol $|M|$ stands for the support of the system M. We write $M \preceq N$ and $N \succeq M$ to mean that M is an elementary subsystem of a system N.

Let us consider a sequence of L-structures A_α, $\alpha \in I$, where I is an ordered set. This sequence is said to be *increasing* if $A_\alpha \subseteq A_\beta$ for any $\alpha < \beta$ in I. By the *union* of an increasing sequence of subsystems A_α $(\alpha \in I)$ we mean an L-structure B whose support is the union of the supports of the structures A_α $(\alpha \in I)$ and which is an extension of each of the systems A_α $(\alpha \in I)$.

One can easily prove the following theorem.

Theorem 3.3. *The union of an increasing sequence of elementary subsystems is an elementary extension of each of these subsystems.*

The main theorems of modern model theory are the following two theorems.

Theorem 3.4 (Löwenheim, Skolem). *Suppose that the cardinality of a signature L does not exceed an infinite cardinal κ. For any infinite L-structure N and every subset A of cardinality κ in the support of the structure N there is an elementary subsystem M of N whose support contains A and is of cardinality κ.*

Theorem 3.5 (Malcev compactness theorem). *Every finitely consistent L-theory is consistent.*

For the proofs of these theorems, see, for instance, [18].

The following statement is an obvious corollary to the Malcev [Mal'tsev] compactness theorem.

Theorem 3.6 (extension theorem). *Every infinite algebraic system has an elementary extension of arbitrarily large cardinality.*

For an L-structure M and an arbitrary subset A of the support of the structure M we say that a finite set $\{\phi_1(x), \dots, \phi_k(x)\}$ consisting of $L(A)$-formulae containing no free variables distinct from x *holds in* $(M, a \mid a \in A)$ if the formula

$$(\exists\, x)(\phi_1(x) \wedge \dots \wedge \phi_k(x))$$

is true in M.

For an L-structure M and an arbitrary subset A of the support of M we say that a set p consisting of $L(A)$-formulae containing no free variables distinct from x is *finitely satisfiable* over A in M if every finite subset $\{\phi_1(x), \dots, \phi_k(x)\}$ of the set p holds in $(M, a \mid a \in A)$.

A finitely satisfiable set p is called a *type* over A in M if for any $L(A)$-formula $\phi(x)$ we have either $\phi \in p$ or $\neg\phi \in p$.

We say that a subset q of type p *isolates* p if p is the only type over A in M that contains q.

Let us consider an L-structure M and an arbitrary subset A of the support of M.

For any $N \succeq M$ and $b \in N$ the set of all $L(A)$-formulae $\phi(x)$ such that $\phi(b)$ is true in

$$(N, a \mid a \in A)$$

forms a type over A in M. Denote this type by $\mathrm{tp}(b/A)$. We say that an element $b \in N$ *realizes* the type $\mathrm{tp}(b/A)$ and that this type is realized in N.

An obvious corollary to the Malcev compactness theorem is the following theorem.

Theorem 3.7. *Consider an L-structure M and an arbitrary subset A of the support of M. For any type p over A in M there is an $N \succeq M$ such that p is realized in N. In other words, there is an $a \in N$ such that $p = \mathrm{tp}(a/A)$.*

Let λ be an infinite cardinal.

Definition 9 (λ-saturated structure). An L-structure M is said to be λ-*saturated* if for any subset A of cardinality less than λ in the support of this structure and for every type p over A this type p is realized in M.

This means that for every subset A of cardinality less than λ in the support of M and every type p in M over A there is an element $a \in M$ such that $p = \mathrm{tp}(a/A)$ in M.

It is clear that every finite L-structure is λ-saturated for every infinite cardinal λ.

Every λ-saturated L-structure of cardinality λ is said to be *saturated*.

Definition 10 (partial isomorphism from A into B). Let A and B be algebraic systems of signature L. Let C be a subset of the support of A and let D be a subset of the support of B. Let τ be a one-to-one map of C onto D. This means that τ is a set of pairs of the form (a, b), where $a \in C$ and $b \in D$, τ contains one and only one pair of the form (a, b) for any $a \in C$, and τ contains one and only one pair of the form (a, b) for any $b \in D$. If $(a, b) \in \tau$, then $\tau(a)$ is b and $\tau^{-1}(b)$ is a.

A map τ of this kind is called a *partial isomorphism* from A into B if:

1) for any relation symbol P in L and any elements $d_1, \dots, d_k$ in C the formula $P^A(d_1, \dots, d_k)$ holds if and only if the formula $P^B(\tau(d_1), \dots, \tau(d_k))$ holds, where k is the arity (the number of argument places) of P (for $k = 0$ this means that P^A coincides with P^B);
2) for any non-0-ary operation symbol f in L with arity k and for any $d_1, \dots, d_k$ in C the element $f^A(d_1, \dots, d_k)$ belongs to C if and only if the element $f^B(\tau(d_1), \dots, \tau(d_k))$ belongs to D, and

$$\tau(f^A(d_1, \dots, d_k)) = f^B(\tau(d_1), \dots, \tau(d_k))$$

   if $f^A(d_1, \dots, d_k)$ belongs to C;
3) for any symbol c of a distinguished element in L we have
   (a) if $c^A \in C$, then $c^B \in D$ and $\tau(c^A) = c^B$,
   (b) if $c^B \in D$, then $c^A \in C$ and $\tau^{-1}(c^B) = c^A$.

A partial isomorphism τ is called an *isomorphism* between A and B if $A = C$ and $B = D$. An isomorphism between A and A is called an *automorphism.* Two structures of the same signature are said to be *isomorphic* if there is an isomorphism between them.

It is easy to see that every two finite L-structures are elementarily equivalent if and only if they are isomorphic.

Theorem 3.8. *For any infinite λ every two elementarily equivalent λ-saturated L-structures A and B of cardinality λ are isomorphic.*

Proof. Let $|A| = \{a_\alpha \mid \alpha < \lambda\}$ and $|B| = \{b_\alpha \mid \alpha < \lambda\}$.

Using induction on α, we construct for $\alpha < \lambda$ some maps τ_α and sets C_α and D_α such that:

(a) τ_α is a one-to-one map of C_α onto D_α;
(b) $\{a_\beta \mid \beta < \alpha\} \subseteq C_\alpha \subseteq |A|$;
(c) $\{b_\beta \mid \beta < \alpha\} \subseteq D_\alpha \subseteq |B|$;
(d) $(A, d \mid d \in C_\alpha) \equiv (B, \tau_\alpha(d) \mid d \in C_\alpha)$.

Here $(A, d \mid d \in C_\alpha)$ and $(B, \tau_\alpha(d) \mid d \in C_\alpha)$ are $L(C_\alpha)$-structures. In $(A, d \mid d \in C_\alpha)$ the value of the name of an element in C_α is the element itself, and in $(B, \tau_\alpha(d) \mid d \in C_\alpha)$ the value of the name of an element d in C_α is $\tau_\alpha(d)$.

We take C_0 and D_0 to be empty. For limit ordinals α we introduce τ_α as the union of all maps τ_β for $\beta < \alpha$. We now assume that the set τ_α has been constructed for $\alpha < \lambda$, and we construct $\tau_{\alpha+1}$.

If a_α is contained in C_α, then we let τ'_α coincide with τ_α and set $b = \tau_\alpha(a_\alpha)$. Otherwise, we consider $p = \mathrm{tp}(a_\alpha/C_\alpha)$. It follows from (d) that p is a type over D_α in B if for any d in C_α the value of the name of the element d is $\tau_\alpha(d)$. Since B is λ-saturated, it follows that this type is realized by some element b. Let $\tau'_\alpha(a_\alpha) = b$ and $\tau'_\alpha(a) = \tau_\alpha(a)$ for $a \in C_\alpha$. Thus, τ'_α defines a one-to-one map of $C_\alpha \cup \{a_\alpha\}$ onto $D_\alpha \cup \{b\}$. If b_α is contained in $D_\alpha \cup \{b\}$, then $\tau_{\alpha+1}$ coincides with τ'_α. Otherwise, we consider a type b_α over $D_\alpha \cup \{b\}$; denote it by p'. It follows from (d) and the choice of b that p' is a type over $C_\alpha \cup \{a_\alpha\}$ in A if the value of the name of an element d in D_α is $\tau_\alpha^{-1}(d)$ and the value of the name of the element b is a_α. Since A is λ-saturated, it follows that this type is realized in A by some element a. We set $\tau_{\alpha+1}(a) = b_\alpha$ and $\tau_{\alpha+1}(d) = \tau'_\alpha(d)$ for $d \in (C_\alpha \cup \{a_\alpha\})$. One can easily see that the conditions (a)–(d) hold for the $\tau_{\alpha+1}$ thus constructed.

It follows from (a), (b), and (c) that τ_λ defines a one-to-one map from $|A|$ onto $|B|$. It easily follows from (d) that this map is an isomorphism between A and B.

Theorem 3.9 (existence of λ-saturated structures). *Suppose that the cardinality of a signature L does not exceed λ and that an infinite L-structure A has cardinality not exceeding 2^λ. Then there is a λ^+-saturated elementary extension B of A with cardinality 2^λ.*

Proof. We construct an increasing sequence of elementary subsystems B_α of length 2^λ such that:

(a) B_0 is equal to A;
(b) for any subset X of cardinality λ in the support of B_α every type over X in B_α is realized in $B_{\alpha+1}$;
(c) for $\alpha > 0$ the cardinality of B_α is equal to 2^λ.

To construct $B_{\alpha+1}$, we find an L-structure B in which all types over all subsets X of cardinality λ in the support of B_α are realized and which is an elementary extension of B_α. The existence of a structure B of this kind follows easily from the Malcev compactness theorem. By the extension theorem, we can assume that the cardinality of B is not less than 2^λ. Since the set of these types has cardinality at most 2^λ, it follows from the Löwenheim–Skolem theorem that the structure B has an elementary subsystem $B_{\alpha+1}$ of cardinality 2^λ in which all types over all subsets X of cardinality λ in the support of B_α are realized and which is an elementary extension of B_α.

Since $\mathrm{cf}(2^\lambda) > \lambda$ (Corollary 2.4), it follows that the union of an increasing sequence of elementary subsystems B_α $(\alpha < 2^\lambda)$ is a λ^+-saturated elementary extension of A which has cardinality 2^λ.

Definition 11 (special structure). An L-structure M of cardinality λ is said to be *special* if M is a union of an increasing sequence of elementary subsystems

$$\{M_\mu \mid \mu \text{ is a cardinal and } \mu < \lambda\}$$

such that the subsystem M_μ is μ^+-saturated for each cardinal μ less than λ. This increasing sequence of elementary subsystems is said to be *specializing* for M. (We recall that μ^+ stands for the least cardinal exceeding μ.) Obviously, every saturated system is special.

The next two theorems are repeatedly used below.

Theorem 3.10. *Elementarily equivalent special L-structures A and B of the same cardinality λ are isomorphic.*

Proof. One must make some refinements in the proof of Theorem 3.8. Let $|A| = \{a_\alpha \mid \alpha < \lambda\}$ and $|B| = \{b_\alpha \mid \alpha < \lambda\}$. Moreover, let

$$\{A_\mu \mid \mu \text{ is a cardinal and } \mu < \lambda\}$$

and

$$\{B_\mu \mid \mu \text{ is a cardinal and } \mu < \lambda\}$$

be specializing sequences for A and B. When constructing τ_α, we can assume in addition that the element $\tau_\alpha(a_\beta)$ is in B_μ if $\beta < \mu$ and that $\tau_\alpha^{-1}(b_\beta)$ is in A_μ if $\beta < \mu$. The remainder of the proof repeats that of Theorem 3.8.

Theorem 3.11. *For any infinite L-structure M and any cardinal λ which is greater than both the cardinality of the signature L and the cardinality of M and satisfies the condition $\lambda^* = \lambda$ there is a special L-structure N of cardinality λ which is an elementary extension of the L-structure M.*

Proof. If λ is not a limit cardinal, $\lambda = \mu^+$, then it follows from the condition $\lambda^* = \lambda$ that $2^\mu = \mu^+$. Then by Theorem 3.9 there is a saturated system N of cardinality λ which is an elementary extension of the L-structure M.

Let us consider the case in which λ is a limit cardinal. In this case, $\mu^+ < \lambda$ and $2^\mu \leqslant \lambda$ for any cardinal μ less than λ.

For a cardinal μ less than λ but not less than κ, where κ is the cardinality of the L-structure M, we construct L-structures N_μ in such a way that the cardinality of N_μ is equal to 2^μ, the structures N_μ are μ^+-saturated, these structures form an increasing sequence of elementary subsystems, and each of these subsystems is an elementary extension of M. For these constructions we use Theorem 3.9. For μ less than κ we take N_κ as N_μ. The union of the sequence

$$\{N_\mu \mid \mu \text{ is a cardinal and } \mu < \lambda\}$$

constructed is the desired special system. This proves Theorem 3.11.

It is easy to see that every restriction of a λ-saturated system is a λ-saturated system. For this reason, every restriction of a special system is a special system.

As was already noted in the Introduction, the relation $<$ on a set I is said to be a relation of *linear order* if the following conditions hold for any elements a, b, and c in I:

(a) $a < b$, $b < a$, or $a = b$ (linearity);
(b) if $a < b$ and $b < c$, then $a < c$ (transitivity);
(c) if $a < b$, then neither of the conditions $a = b$ and $b < a$ holds (antisymmetry).

A linear order $(I, <)$ is said to be *dense* if for any $a < b$ in I there is an element c in I such that $a < c < b$.

A linear order $(I, <)$ is said to be *complete* if, whenever I is partitioned into two non-empty subsets such that every element of the first is less than every element of the second and every element of I belongs to one of these two subsets, either there is a greatest element of the first subset or there is a least element of the second subset.

Definition 12 (indiscernible sequence). A subset I of the support of an L-structure M linearly ordered by the relation $<$ is called a *θ-indiscernible sequence* in M with respect to the relation $<$ for an L-formula $\theta(x_1, \dots, x_n)$ if for any two n-tuples $\overline{a}$ and $\overline{b}$ of elements in I such that $a_1 < \cdots < a_n$ and $b_1 < \cdots < b_n$ the statement $\theta(\overline{a})$ is true in M if and only if $\theta(\overline{b})$ is true in M.

Let a subset I of the support of an L-structure M be linearly ordered by the relation $<$. The subset I is called an *indiscernible sequence* in M with respect to the relation $<$ if the set I is a θ-indiscernible sequence for any natural number n and any L-formula $\theta(x_1, \dots, x_n)$.

An indiscernible sequence I is said to be *effective* if there is an algorithm determining whether or not the formula $\theta(\overline{a})$ is true in M for any L-formula $\theta(x_1, \dots, x_n)$ and any $a_1 < \cdots < a_n$ in I.

An immediate corollary to the Ramsey theorem (Theorem 2.6) is as follows.

Theorem 3.12. *Let M be an infinite L-structure and let $<$ be a linear ordering of the universe of M. For any L-formula $\theta(x_1, \dots, x_n)$ there is a set I which is both an infinite subset of the universe of the structure M and a θ-indiscernible sequence in M with respect to $<$.*

This theorem, together with the Malcev compactness theorem (Theorem 3.5), easily implies the following result.

Theorem 3.13 (existence of an indiscernible sequence). *For an arbitrary infinite L-structure M linearly ordered by a relation $<$ with name appearing in L whose support has no elements in common with a set I and for any linear order $(I, <)$ there is an elementary extension N of M such that the support of the L-structure N contains I, I is an indiscernible sequence in N with respect to $<$, and $(I, <)$ is a subsystem of the $\{<\}$-restriction of the L-structure N.*

4. Definitions

By the *universe* of a signature L we mean an arbitrary infinite algebraic system of signature L. For brevity of notation, for a given universe U we often denote by U the support of the universe U as well. We assume as usual that the signature of the universe under consideration is finite. As was already mentioned in the Introduction, we consider only *linearly ordered* universes. This term is used for universes with basic relations containing a binary relation which is a linear order relation (that is, a linear, transitive, and antisymmetric relation). We use the symbol $<$ as the name of this linear order.

By a *database scheme* we mean a finite family of names of relations and distinguished elements equipped with an indication of the arity of every relation. All relations with names in the database scheme under consideration are assumed to be of finite arity. Thus, a database scheme is another signature which contains no operation symbols other than symbols of distinguished elements. We always assume that a database scheme has no symbols in common with the signature of the universe.

We denote by (L, τ) the signature obtained by adding symbols of the signature τ to the symbols of the signature L, where the arity of any symbol remains unchanged.

If U is a given universe, then by a *state* of a database scheme ρ, or a ρ-state, or a database with a scheme ρ we mean a map which, to each name of a relation of arity k in the scheme ρ, assigns a specific relation of arity k on U, which can be regarded as a subset of the set U^k of k-tuples of elements of the universe and can be given as a table with k columns and some set of rows; moreover, to each name of a distinguished element this map assigns the value given by a specific element of the universe. A state s enriches the universe U of the signature L to an (L, ρ)-structure, which we denote by (U, s).

The *active domain* of a state s is the set of all elements of all rows of all tables in s and the values of all symbols of the distinguished elements for this state. The active domain of a state s is denoted by $\mathrm{AD}(s)$. A state s is said to be *finite* if its active domain is a finite set; in other words, the relations in s are finite families of sequences (of the corresponding length) of elements of the universe and can be given by finite tables.

A state s is said to be a *state over I* if $\mathrm{AD}(s) \subseteq I$, that is, corresponding to every name of a relation in the database scheme under consideration is a relation of the same arity on the set I, and the value of every symbol of a distinguished element in the database scheme is an element of the set I.

For the query language we use the language of first-order predicate logic. This means that the queries are formulae of this language. In the formulae we can use either just the names of the scheme ρ under consideration and the symbol $<$ of the order relation, or also all the other operation and relation symbols in the signature L of the universe. Formulae of the first kind are said to be *restricted* ρ-queries and formulae of the second kind are said to be *extended* ρ-queries.

Definition 13 (restricted and extended queries). The *restricted* ρ-queries are the $(<,\rho)$-formulae and the *extended* ρ-queries are the (L,ρ)-formulae.

If a formula defining a query is closed (contains no free variables), then the query is said to be *Boolean.*

Theorem 4.1. *Let ρ consist of a symbol P which is a unary relation symbol. No Boolean restricted ρ-query is true for a given ρ-state if and only if the active domain of this state contains an even number of elements.*

Proof. The proof proceeds by contradiction. Let such a query exist and let it be defined by a closed $(<,\rho)$-formula ϕ. We can assume that ϕ is a prenex formula. Let ϕ contain k quantifiers. Suppose that ϕ is of the form

$$(Q_1x_1)\dots(Q_kx_k)\psi(x_1,\dots,x_k),$$

where $(Q_1x_1)\dots(Q_kx_k)$ is a quantifier prefix.

We introduce the notion of distance between elements of the universe in a given subset P of the universe. The distance from a to a in any subset of the universe is equal to 0. Let a and b be distinct and let $a<b$. By the distance from a to b in a subset P of the universe we mean 1 plus the number of elements of the universe in P between a and b. We denote this distance by $\rho^P(a,b)$. If the subset is not indicated, then it is assumed to be the entire universe. If the subset is infinite, then the distance can also be infinite. If $b<a$, then $\rho(a,b)=-\rho(b,a)$.

Let A be a subset (of the universe) which contains 2^{2k+2} elements such that the absolute value of the distance between any two distinct elements of A is greater than 2^{2k+2} and the absolute value of the distance from each of the elements to any end of the universe (if it has at least one end) is greater than 2^{2k+2}. For example, if the universe is densely ordered and has no ends, then we can take an arbitrary subset A (of the universe) containing 2^{2k+2} elements. Let a state s_1 assign the entire subset A to P and let a state s_2 assign to P the set B obtained from A by deleting a single non-maximal and non-minimal element.

It is easy to see that these states are not distinguished by the formula ϕ. However, the number of elements of P is even in the first case and odd in the second case. For this reason, the formula ϕ does not distinguish states with even and odd number of elements in the active domain.

For the proof we establish a more general statement by backwards induction on i. To formulate this statement, we introduce the following definitions.

By an *i-enrichment* of the state s_1 we mean a state which assigns the element a_j to the symbol c_j for any $j=1,\dots,i$. By an *i-enrichment* of the state s_2 we mean a state which assigns the element b_j to the symbol c_j for any $j=1,\dots,i$.

We refer to the elements $a_1,\dots,a_i$, to the ends of the universe, and to the ends of A as *distinguished* elements. Moreover, if a is an arbitrary distinguished element not belonging to A, then the elements of A which are the nearest to it are also regarded as distinguished elements. The image of an end of A or of an end of the universe is this end itself. For the image of an element a_j in $a_1,\dots,a_i$ we take b_j.

If a_* and a^* are the elements of A nearest from the left and from the right to a distinguished element $a \notin A$, then their images are the elements b_* and b^* of A nearest from the left and from the right to the image of a. The image of an interval determined by two distinguished elements is the interval determined by the images of these distinguished elements. The image of a distinguished element a is denoted by $s(a)$. Let $\varepsilon_i = 2^{k+1-i} - 1$.

Lemma 4.2. *Let i-enrichments of the states s_1 and s_2 satisfy the following conditions for any elements a' and a'' distinguished in s_1 and for their images b' and b'' distinguished in s_2 :*
1) $|\rho(a', a'')| \geqslant \varepsilon_i$ *if and only if* $|\rho(b', b'')| \geqslant \varepsilon_i$;
2) $|\rho^A(a', a'')| \geqslant \varepsilon_i$ *if and only if* $|\rho^B(b', b'')| \geqslant \varepsilon_i$;
3) *if* $|\rho(a', a'')| < \varepsilon_i$, *then* $\rho(b', b'') = \rho(a', a'')$;
4) *if* $|\rho^A(a', a'')| < \varepsilon_i$, *then* $\rho^B(b', b'') = \rho^A(a', a'')$;
5) $a' < a''$ *if and only if* $b' < b''$;
6) $a' \in A$ *if and only if* $b' \in B$.
In this case the i-enrichments of the states s_1 and s_2 are not distinguished by the formula

$$(Q_{i+1}x_{i+1}) \dots (Q_k x_k)\psi(c_1, \dots, c_i, x_{i+1}, \dots, x_k). \tag{2}$$

Proof of the lemma. The lemma is proved by backwards induction on i. For $i = k$ it follows from the conditions 5) and 6) that the k-enrichments of the states s_1 and s_2 cannot be distinguished by the quantifier-free formula

$$\psi(c_1, \dots, c_i, c_{i+1}, \dots, c_k).$$

Therefore, the lemma is valid for $i = k$.

Let $i < k$ and let the lemma hold for $i + 1$. Let the conditions 1)–6) hold for i.

We must prove that either (2) is true both for the i-enrichment of s_1 and for the i-enrichment of s_2 or (2) is false both for the i-enrichment of s_1 and for the i-enrichment of s_2. Suppose the contrary. In this case the formula holds for one of these states. We prove that this formula holds true also for the other state, which will complete the proof of the lemma.

Let (2) hold true for the i-enrichment of s_1. If this formula holds true for the other state, then the argument is quite similar.

Case 1. Let Q_{i+1} be the existential quantifier $\exists$. We choose an element a_{i+1} such that if $s_1(c_{i+1}) = a_{i+1}$ for the corresponding $(i+1)$-enrichment of the state s_1, then the formula

$$(Q_{i+2}x_{i+2}) \dots (Q_k x_k)\psi(c_1, \dots, c_i, c_{i+1}, x_{i+2}, \dots, x_k) \tag{3}$$

holds true.

Let us now choose an element b_{i+1} in such a way that the conditions 1)–6) hold for $s_2(c_{i+1}) = b_{i+1}$ with i replaced by $i + 1$. If such a choice is possible, then, by the induction assumption, the $(i + 1)$-enrichments of the states s_1 and s_2 are not distinguished by the formula (3), and the formula (2) remains valid for the i-enrichment of s_2 as well.

If a_{i+1} coincides with one of the distinguished elements, then we assume that b_{i+1} coincides with the image of this element. In this case it is obvious that one can satisfy the conditions 1)–6) for $s_2(c_{i+1}) = b_{i+1}$ with i replaced by $i + 1$.

Below we assume that the element a_{i+1} differs from any distinguished element.

The distinguished elements partition the universe into finitely many intervals, and a_{i+1} belongs to one of these intervals. Let us choose a b_{i+1} in the corresponding interval. If these rules are satisfied, then the condition 5) certainly holds.

Let a_* and a^* be neighbouring distinguished elements with a_{i+1} between them. We do not treat separately the case in which a_{i+1} is taken to be less than the least distinguished element or greater than the greatest distinguished element. These cases can be treated in a similar way.

Case 1.1. $a_{i+1} \in A$.

Case 1.1.1. If $\rho^A(a_*, a_{i+1})$ or $\rho^A(a_{i+1}, a^*)$ is less than ε_{i+1}, then for b_{i+1} we take an element of B at the same distance in B from $s(a_*)$ or $s(a^*)$, respectively.

If the element a_{i+1} here is close in A to some distinguished element in s_1, for instance, to a, then if $\rho^A(a_*, a_{i+1}) < \varepsilon_{i+1}$, we see that

$$\begin{aligned}|\rho^A(a, a_*)| &= |\rho^A(a, a_{i+1}) + \rho^A(a_{i+1}, a_*)| \\ &\leqslant |\rho^A(a, a_{i+1})| + |\rho^A(a_{i+1}, a_*)| < 2\varepsilon_{i+1} < \varepsilon_i.\end{aligned}$$

Then by 2) and 4), we obtain $\rho^B(s(a), s(b_*)) = \rho^A(a, a_*)$ and

$$\begin{aligned}\rho^B(s(a), b_{i+1}) &= \rho^B(s(a), s(a_*)) + \rho^B(s(a_*), b_{i+1}) \\ &= \rho^A(a, a_*) + \rho^A(a_*, a_{i+1}) = \rho^A(a, a_{i+1}).\end{aligned}$$

Let us now consider the distance between the elements a_{i+1} and a_* and also the distance between their images in the universe. Obviously, it suffices to consider only the case in which $a_* \notin A$, because otherwise $\rho(a_*, a_{i+1}) \geqslant 2^{2k+2}$, and the same inequality holds for the images. If $\rho^A(a_*, a_{i+1}) \geqslant 2$, then $\rho(a_*, a_{i+1}) \geqslant 2^{2k+2} \geqslant \varepsilon_i$, and the same holds for the images. Let $\rho^A(a_*, a_{i+1}) = 1$. In this case the element a_{i+1} would be distinguished, as an element of A nearest to a_*.

Case 1.1.2. If both the distances $\rho^A(a_*, a_{i+1})$ and $\rho^A(a_{i+1}, a^*)$ are greater than or equal to ε_{i+1}, then we obviously have $\rho^A(a_*, a^*) \geqslant 2\varepsilon_{i+1}$. Then

$$\rho^B(s(a_*), s(a^*)) \geqslant 2\varepsilon_{i+1}$$

by 2) and 4), and there is an element of B whose distance both from $s(a_*)$ and from $s(a^*)$ is not less than ε_{i+1}; one must take this element as b_{i+1}.

Case 1.2. $a_{i+1} \notin A$.

Case 1.2.1. If $\rho(a_*, a_{i+1})$ or $\rho(a_{i+1}, a^*)$ is less than ε_{i+1}, then as b_{i+1} we take an element of the universe at the same distance from $s(a_*)$ or $s(a^*)$, respectively. The other considerations are the same as in Case 1.1.1 with ρ^A and ρ^B replaced by ρ.

Case 1.2.2. Otherwise, let a_{**} (a^{**}, respectively) be the element of A nearest to a_{i+1} from the left (from the right, respectively). If at least one of the distances $\rho^A(a_*, a_{**})$ or $\rho^A(a^{**}, a^*)$ is less than ε_{i+1}, then we construct $s(a_{**})$ and $s(a^{**})$ as in Case 1.1.1. If both the distances are greater than or equal to ε_{i+1}, then $\rho^A(a_*, a^*) \geqslant 2\varepsilon_{i+1} + 1 = \varepsilon_i$. In this case one can also find two consecutive elements of B located between $s(a_*)$ and $s(a^*)$ and such that the distances from these elements to $s(a_*)$ and $s(a^*)$ in B are not less than ε_{i+1}. We take these elements as $s(a_{**})$ and $s(a^{**})$.

We can now assume that there are no elements of A between a_{i+1} and the nearest distinguished elements. If Case 1.2.1 is applicable after this step, then we proceed as in Case 1.2.1. Otherwise both the distances to the nearest distinguished elements are greater than ε_{i+1}. In this case the corresponding distance between the images of the nearest distinguished elements in s_2 is greater than ε_i, and one can find an element in s_2 which is sufficiently far from the images of these distinguished elements and is located between these images.

Case 2. Q_{i+1} is the universal quantifier $\forall$, and the formula (2) is false for the i-enrichment of the state s_2. Let us choose an element b_{i+1} such that the formula (3) is false for $s_2(c_{i+1}) = b_{i+1}$ for the $(i+1)$-enrichment of the state s_2. Arguing as above in Case 1, we choose an element a_{i+1} such that the formula (3) is false for $s_1(c_{i+1}) = a_{i+1}$ for the $(i+1)$-enrichment of the state s_1. However, this contradicts the assumption that the formula (2) is true for the i-enrichment of s_1. This completes the proof of Lemma 4.2.

It follows from Lemma 4.2 for $i = 0$ that the states s_1 and s_2 are not distinguished by the formula ϕ. This proves Theorem 4.1.

We sometimes consider partial isomorphisms of restrictions of the systems under consideration. In this case it is important to note what the restricted signature is for which the map under consideration is a partial isomorphism. To this end, we use the notion of partial L_1-isomorphism. This means that we study the restrictions of the structures to systems of signature L_1 and partial isomorphisms between these systems. In particular, any partial $<$-isomorphism is a partial isomorphism of $<$-restrictions of the structures.

In more detail, a one-to-one map f is said to be a *partial $<$-isomorphism* of a subset X of the support of a system U into U and is denoted by $f\colon X \to U$ if f is a one-to-one map of X into the support of the system U and for any x and y in X the relation $x < y$ holds in U if and only if $f(x) < f(y)$ in U. If X coincides with the support of the system U, then f is called an *$<$-automorphism* of the system U.

We consider only formulae of a special form which give the so-called *locally generic* queries.

Definition 14 (locally generic query). A query $\Phi(\overline{x})$ is said to be *locally generic* for the states of class K (over the states of class K, or with respect to the states of class K) if

$$(U, s) \models \Phi(\overline{a}) \;\Leftrightarrow\; (U, f(s)) \models \Phi(f(\overline{a}))$$

for any partial $<$-isomorphism $f\colon X \to U$ such that $X \subseteq U$, for any state s over X of class K such that $f(s)$ belongs to K, and for any sequence $\overline{a}$ of elements of X.

Here the image of a sequence is the sequence of images, and the image of a table is the table formed by the images of the elements in the given table. The image of a state (of a finite family of tables) is the family of images of these tables. We recall that if a state s is a state over X, then $\mathrm{AD}(s) \subseteq X$. For this reason, the value $f(s)$ is well defined.

If K is the class of all finite states, then we speak of the local genericity of queries for finite states (or simply of local genericity).

Definition 15 (generic query). A query $\Phi(\overline{x})$ is said to be *generic* for the states of class K (or with respect to the states of class K) if

$$(U,s) \models \Phi(\overline{a}) \Leftrightarrow (U, f(s)) \models \Phi(f(\overline{a}))$$

for any $<$-automorphism f of the universe U and for any state s of class K such that $f(s)$ belongs to K.

For brevity, a closed L-formula is called an *L-sentence*.

For an (L,ρ)-sentence ψ and any natural number m one can easily construct an L-sentence ψ_m such that for any L-structure V the sentence ψ_m holds in V if and only if ψ holds for all ρ-states over V whose active domain contains at most m elements.

We include an (L,ρ)-sentence ψ in the set $\mathrm{Fin}(V,\rho)$ if and only if ψ_m belongs to $\mathrm{Th}(V)$ for any natural m. It is clear that if $W \equiv V$, then

$$\mathrm{Fin}(V,\rho) = \mathrm{Fin}(W,\rho).$$

For this reason, the (L,ρ)-theory $\mathrm{Fin}(T,\rho)$ is well defined as $\mathrm{Fin}(W,\rho)$, where W is an arbitrary model of a complete L-theory T.

Definition 16 (pseudo-finite state). A ρ-state s for an L-structure W is said to be *pseudo-finite* in W if (W,s) is a model of the (L,ρ)-theory $\mathrm{Fin}(W,\rho)$.

5. Criterion for the collapsability of an extended query to a restricted query

The following theorem is taken from [3]. Here and below, the notation $\phi \leftrightarrow \psi$ is an abbreviation for $(\phi \to \psi) \wedge (\psi \to \phi)$.

Theorem 5.1. *The following three conditions are equivalent for any universe U and any extended Boolean ρ-query ϕ:*
1) *there is a restricted ρ-query ψ equivalent in U to the query ϕ for the finite states of the database;*
2) *the query ϕ is generic for any pseudo-finite states in V for any $V \equiv U$;*
3) *for some uncountable cardinality κ such that $\kappa = \kappa^*$ the query ϕ is generic for any pseudo-finite states in V for the special system $V \equiv U$ of cardinality κ.*

Proof. 1)⇒2). Suppose that ϕ is equivalent in U to a restricted query ψ for the finite states. In this case $\phi \leftrightarrow \psi$ belongs to $\mathrm{Fin}(U,\rho)$, and hence to $\mathrm{Fin}(V,\rho)$ for any $V \equiv U$. Since every restricted query is generic with respect to any states, it follows that ψ is generic with respect to any states of V. Therefore, ϕ is generic with respect to the pseudo-finite states of V.

2)⇒3) obviously holds.

3)⇒1). Let $T = \mathrm{Th}(U)$.

Let

$$\rho = \{R_1, \dots, R_n, c_1, \dots, c_k\}.$$

We choose

$$\rho' = \{R'_1, \dots, R'_n, c'_1, \dots, c'_k\}$$

as a copy of the scheme ρ, assuming that the schemes ρ and ρ' have no elements in common and the arity of R_i and R_i' is the same for any $i \in \{1, \dots, n\}$. For an (L, ρ)-sentence θ we denote by $\theta(\rho')$ the (L, ρ')-copy of θ. In more detail, $\theta(\rho')$ can be obtained from θ by replacing every occurrence of R_i and c_j by the occurrence of R_i' or c_j', respectively. In particular, $\theta(\rho)$ coincides with θ. Let $\overline{\rho} = \rho \cup \rho'$.

We note first that the condition 3) implies the inconsistency of the following theory Γ:

$$\mathrm{Fin}(T, \overline{\rho}) \cup \{\theta(\rho) \leftrightarrow \theta(\rho') : \theta \text{ is a } (<, \rho)\text{-sentence}\} \cup \{\phi(\rho), \neg\phi(\rho')\}.$$

Suppose the contrary. Let (W, r, r') be a model of Γ. Using the Löwenheim–Skolem theorem and Theorem 3.11, one can assume that this model is special and is of cardinality κ.

In this case the restrictions W, $(W \restriction \{<\}, r)$, and $(W \restriction \{<\}, r')$ of this model are also special systems. Therefore, W and V are elementarily equivalent special systems of the same cardinality. By Theorem 3.10, W and V are isomorphic.

It also follows from Γ that the systems $(W \restriction \{<\}, r)$ and $(W \restriction \{<\}, r')$ are elementarily equivalent. They also have the same cardinality. Therefore, these systems are also isomorphic. Thus, there is an $<$-automorphism of the system W which takes r to r'. Since (r, r') is a pseudo-finite state in W, $\phi(\rho)$ is true in (W, r), and $\phi(\rho')$ is false in (W, r'), it follows that the query ϕ is not generic for the pseudo-finite states in W. Since V and W are isomorphic, this contradicts the condition 3).

By the Malcev compactness theorem (Theorem 3.5), Γ has a finite inconsistent part. This means that for some $m < \omega$ and some $(<, \rho)$-sentences $\theta_0, \dots, \theta_{m-1}$ we have

$$\mathrm{Fin}(T, \overline{\rho}), \qquad \bigwedge_{i<m} (\theta_i(\rho) \leftrightarrow \theta_i(\rho')), \qquad \phi(\rho) \vdash \phi(\rho')$$

(if all the left-hand formulae are true in some $(L, \overline{\rho})$-structure, then the right-hand formula is also true in this structure). Let $\theta^1 = \theta$ and $\theta^0 = \neg\theta$. Then obviously for some $\tau_{ij} \in \{0, 1\}$ the extended query ϕ is equivalent, with respect to the finite states of U, to the restricted query

$$\bigvee_{i<l} \bigwedge_{j<m} \theta_i^{\tau_{ij}}.$$

6. Relative pseudo-finite homogeneity and isolation properties

The material of this section is taken from [11].

Theorem 6.1. *Let U be an arbitrary universe of signature L and let an extended Boolean ρ-query ϕ be locally generic for the finite states over U.*

The following two conditions are equivalent:

1) *for some uncountable cardinality κ such that $\kappa = \kappa^*$ there is a special model (V, J) of cardinality κ such that J is an indiscernible sequence in V, $V \equiv U$, and the query ϕ is locally generic with respect to the pseudo-finite states over J in V;*
2) *ϕ is equivalent in U for finite states over U to some restricted ρ-formula.*

Proof. Let κ and (V, J) satisfy the assumptions of the theorem. This means that κ is an uncountable cardinality satisfying the condition $\kappa = \kappa^*$, the special model (V, J) is of cardinality κ, and J is an indiscernible sequence in V, where $V \equiv U$.

Lemma 6.2. *Let a partial $<$-isomorphism g of V transform a pseudo-finite state p over J in V of the scheme ρ into a pseudo-finite state p' over J in V of the same scheme. Then g can be extended to an $<$-automorphism h of the system $V \restriction \{<\}$.*

Proof. We note that for any $a \in J$ the set

$$\{x \in V \mid x < a \text{ in } V\}$$

is of cardinality κ. Indeed, let V_α $(\alpha < \kappa)$ be a specializing sequence for V. Then there is a cardinal $\beta < \kappa$ such that $a \in V_\alpha$ for any α with $\beta < \alpha < \kappa$. Since V_α is α^+-saturated, it follows that the cardinality of the set

$$\{x \in V_\alpha \mid x < a \text{ in } V_\alpha\}$$

is not less than α.

One can prove similarly that for any a and b with $a < b \in J$ the sets

$$\{x \in V \mid a < x < b \text{ in } V\}$$

and

$$\{x \in V \mid a < x \text{ in } V\}$$

are of cardinality κ.

Since the restriction of any special system is a special system and since J is an indiscernible sequence in V, it follows that for $a, b \in J$ the substructures

$$(\{x \in V \mid x < a \text{ in } V\}, <)$$

and

$$(\{x \in V \mid x < b \text{ in } V\}, <)$$

of the structure $V \restriction \{<\}$ are elementarily equivalent special structures of the same cardinality.

Therefore, these structures are isomorphic. Suppose that $h_{a,b}$ maps the substructure

$$(\{x \in V \mid x < a \text{ in } V\}, <)$$

isomorphically onto the substructure

$$(\{x \in V \mid x < b \text{ in } V\}, <).$$

Similarly, for $a, b \in J$ the substructures

$$(\{x \in V \mid a < x \text{ in } V\}, <)$$

and

$$(\{x \in V \mid b < x \text{ in } V\}, <)$$

of the structure $V \upharpoonright \{<\}$ are elementarily equivalent special structures of the same cardinality. Therefore, these structures are isomorphic. Let $g_{a,b}$ map the substructure

$$(\{x \in V \mid a < x \text{ in } V\}, <)$$

isomorphically onto the substructure

$$(\{x \in V \mid b < x \text{ in } V\}, <).$$

Similarly, for $a < b$ in J and $c < d$ in J the substructures

$$(\{x \in V \mid a < x < b \text{ in } V\}, <)$$

and

$$(\{x \in V \mid c < x < d \text{ in } V\}, <)$$

of the structure $V \upharpoonright \{<\}$ are also isomorphic. Let $h_{a,b;c,d}$ map the substructure

$$(\{x \in V \mid a < x < b \text{ in } V\}, <)$$

isomorphically onto the substructure

$$(\{x \in V \mid c < x < d \text{ in } V\}, <).$$

We can now construct an $<$-automorphism h of $V \upharpoonright \{<\}$ that extends g. Let u (v) be the least (greatest) element of the active domain of the state p. For an arbitrary $x \in V$ not in the active domain of the state p we have either one of the relations $x < u$ and $v < x$ or there are elements ${}_x a$ and a_x in the active domain of p such that ${}_x a < x < a_x$ and there are no elements in the active domain of p between ${}_x a$ and a_x. If $x < u$, then we set $h(x) = h_{u,g(u)}(x)$. If $v < x$, then we set $h(x) = g_{v,g(v)}(x)$. Otherwise, we set $h(x) = h_{{}_x a, a_x; g({}_x a), g(a_x)}(x)$. This completes the proof of Lemma 6.2.

Let the query ϕ be equivalent in U for finite states over U to some restricted ρ-formula.

Since ϕ is equivalent in U for finite states over U to some closed formula of the signature $(<, \rho)$, it follows that ϕ is equivalent in V for pseudo-finite states over V to some closed formula of the signature $(<, \rho)$, and hence the query ϕ is preserved under the isomorphisms of the structure $(V \upharpoonright \{<\}, p)$ that are $<$-automorphisms transforming p to pseudo-finite states of V. Since h in Lemma 6.2 is just an $<$-automorphism of the structure $(V \upharpoonright \{<\}, p)$ transforming p to p', it follows that ϕ is true in (V, p) if and only if ϕ is true in (V, p'). This means that the query ϕ is locally generic with respect to pseudo-finite states over J in V.

We assume now that a query ϕ is locally generic with respect to pseudo-finite states over J in V.

Suppose that ϕ is not equivalent in U to any restricted ρ-formula for finite states over U.

By Theorem 5.1, in this case there exist a model $(W, I) \equiv (V, J)$ and a pseudo-finite $\overline{\rho}$-state (p, p') over (W, I) such that some $<$-automorphism h of the model W transforms p to p' in such a way that $(W, p) \models \phi(\rho)$ and $(W, p') \models \neg\phi(\rho')$. One can assume that (W, I, p, p') is a special model of cardinality κ. However, in this case $(W, I) = (V, J)$.

Let ρ', σ, and σ' be copies of ρ such that the schemes ρ, ρ', σ, and σ' are pairwise disjoint. Let

$$\eta = \rho \cup \rho' \cup \sigma \cup \sigma' \cup \{F, F'\},$$

where F and F' are new symbols of binary relations.

We denote the (L, P)-theory of the structure (V, J) by T. We are going to prove the consistency of the family Γ of (L, P, η)-sentences, where Γ claims the existence, for some model of the (L, P)-theory T, of an η-state $(r, r', s, s', F_0, F_0')$ such that:

1) (r, r') satisfies $\mathrm{Th}(V, J, p, p')$;
2) $(r, r', s, s', F_0, F_0')$ satisfies $\mathrm{Fin}(T, \eta)$;
3) F_0 and F_0' are partial $<$-isomorphisms which transform r to s and r' to s', respectively;
4) s and s' are states over P;
5) s satisfies $\phi(\sigma)$ and s' satisfies $\neg\phi(\sigma')$.

Suppose that Γ is consistent. Let $(W_1, J_1, r_1, r_1', s_1, s_1', F_1, F_1')$ be a special model of cardinality κ for Γ. Then (W_1, J_1) is a special model of cardinality κ. Thus, we can assume that $(W_1, J_1) = (V, J)$. It follows from 1) that (W_1, r_1, r_1') and (V, p, p') are isomorphic. Let an $<$-automorphism h_1 of the structure V transform r_1 to r_1'. It follows from 2) that the η-state $(r_1, r_1', s_1, s_1', F_1, F_1')$ is pseudo-finite in (V, J). It follows from 3) that the partial $<$-isomorphism $g = F_1' \circ h_1 \circ F_1^{-1}$ transforms s_1 to s_1'. It follows from 4) that s_1 and s_1' are states over J. It follows from 5) that s_1 satisfies $\phi(\sigma)$ and s_1' does not satisfy $\phi(\sigma')$. This means that ϕ is not locally generic for pseudo-finite states over J in V, which contradicts the choice of (V, J).

By the Malcev compactness theorem, it remains to prove the finite consistency of Γ. Let us take any element $\gamma \in \mathrm{Th}(V, J, p, p')$. It suffices to find a finite η-state (r, r', s, s', F, F') over V that satisfies the conditions γ, 3), 4), and 5).

Since (p, p') is pseudo-finite over (V, J), there is a finite $\overline{\rho}$-state (r, r') over V satisfying the condition $\gamma \wedge \phi(\rho) \wedge \neg\phi(\rho')$. It is a very simple task to find s, s', F, and F' satisfying 3) and 4). Since ϕ is locally generic for finite states, it follows that the condition 5) is satisfied. This proves the finite consistency of Γ and completes the proof of the theorem.

Let A and B be algebraic systems of signature L. Let C be a subset of the support of the system A and D a subset of the support of the system B. Let h be a one-to-one map of C onto D. We denote by $(B, h(a) \mid a \in C)$ an enrichment of the L-structure B to an $L(C)$-structure such that for any $a \in C$ the value of the added name of the element a is $h(a)$. The map h is said to be an *elementary map* from A to B if $(B, h(a) \mid a \in C)$ and $(A, a \mid a \in C)$ are elementarily equivalent as algebraic systems of signature $L(C)$. If A coincides with B, then h is called an elementary map into B.

Definition 17. A complete theory T has the *first pseudo-finite homogeneity property* if there exist a model M of the complete theory T and an infinite set I which is an indiscernible sequence in M such that

> for any structure (N, J) elementarily equivalent to the structure (M, I), for any pseudo-finite subsets A and B of the set J in the model N, for any finite subsets C and D of N, for any map h which is elementary in N and transforms $(A \cup C)$ one-to-one onto $(B \cup D)$ with ω-saturated (N, A, B, h), and for any $a \in N$ there is an element $b \in N$ such that $h \cup \{(a, b)\}$ is an elementary map in N.

Definition 18. A complete theory T has the *second pseudo-finite homogeneity property* if there exist a model M of the complete theory T and an infinite set I which is an indiscernible sequence in (M, I) such that

> for any structure (N, J) elementarily equivalent to the structure (M, I), for any pseudo-finite subsets A and B of the set J in the model N, for any finite subsets C and D of N, for any map h which is elementary in (N, J) and transforms $(A \cup C)$ one-to-one onto $(B \cup D)$ with ω-saturated (N, J, A, B, h), and for any $a \in N$ there is an element $b \in N$ such that $h \cup \{(a, b)\}$ is an elementary map in (N, J).

Theorem 6.3. *Suppose that the theory of a universe U has the first (second) pseudo-finite homogeneity property. Then every extended query ϕ locally generic for finite states over U is equivalent for finite states over U to some restricted query.*

Proof. The proof follows that of Theorem 5.4 in [3]. It suffices to show that ϕ satisfies the assumption of Theorem 6.1.

Let $\kappa = \kappa^* > \omega$. Let $(V, J) \equiv (M, I)$ and (V, J) be a special model of cardinality κ.

We also consider the pseudo-finite ρ-states r and r' over J in V such that r is transformed to r' by a partial $<$-isomorphism g in V whose domain is a set A pseudo-finite in V which is the active domain of the state r, and the set of values of g is a pseudo-finite set A' which is the active domain of the state r'.

We must prove that ϕ holds in (V, r) if and only if ϕ holds in (V, r').

One can assume here that (V, J, A, A', g) is ω-saturated. Indeed, (V, J, A, A', g) is ω-saturated if the model (V, J, r, r', g) is ω-saturated. Let us consider a special model

$$(V_0, I_0, r_0, r'_0, g_0),$$

of cardinality κ and elementarily equivalent to (V, J, r, r', g). It suffices to prove the statement for $(V_0, I_0, r_0, r'_0, g_0)$. The last model is ω-saturated, because $\mathrm{cf}(\kappa) \geqslant \omega$.

It suffices to show that g is an (L, ρ)-elementary map from (V, r) to (V, r').

However, thanks to the L-indiscernibility of J (the (L, P)-indiscernibility of J), the map g certainly is an L-elementary ((L, P)-elementary) map, and moreover, it is a partial (L, ρ)-isomorphism. For this reason, the truth value of every quantifier-free formula of the signature $(L, \rho)(A)$ (of the signature $(L, P, \rho)(A)$) in $(V, r, a \mid a \in A)$ (in $(V, J, r, a \mid a \in A)$) coincides with the truth value of the same formula in $(V, r', g(a) \mid a \in A)$ (in $(V, J, r', g(a) \mid a \in A)$, respectively).

Suppose that the truth value of every prenex formula with a smaller number of quantifiers of signature $(L, \rho)(A)$ (of signature $(L, P, \rho)(A)$) in

$$(V, r, a \mid a \in A)$$

(in $(V, J, r, a \mid a \in A)$) coincides with the truth value of the formula in

$$(V, r', g(a) \mid a \in A)$$

(in $(V, J, r', g(a) \mid a \in A)$, respectively).

We must prove that this statement holds for prenex formulae with a given number of quantifiers.

For the proof it suffices to assume for a finite C_i that

$$g_i \colon (A \cup C_i) \to (A' \cup C_i')$$

extends g and is an L-elementary ((L, P)-elementary) map, choose an arbitrary $c \in V$, and find a $c' \in V$ such that if we set $g_{i+1} = g_i \cup \{(c, c')\}$, then the map g_{i+1} turns out to be L-elementary ((L, P)-elementary, respectively).

However, the existence of the element c' follows from the definition of pseudo-finite homogeneity, the fact that the active domain of every pseudo-finite state is a pseudo-finite set, and the fact that the enrichment of an ω-saturated structure by finitely many distinguished elements is again an ω-saturated structure.

We claim that the active domain of every pseudo-finite state is a pseudo-finite set (this fact was noted in the proof of Theorem 5.4 in [3]).

Let us consider the database scheme $\tau = \{P\}$ in which P is a unary relation symbol. For any (L, τ)-sentence γ and any ρ-state s one can easily construct an (L, ρ)-sentence γ^* for which $(V, s) \models \gamma^*$ if and only if $(V, \mathrm{AD}(s)) \models \gamma$.

Now let s be a pseudo-finite ρ-state in V and let $\gamma \in \mathrm{Fin}(V, \tau)$. Since the active domain of any finite state is finite, it follows that $(V, r) \models \gamma^*$ for any finite ρ-state r. Therefore, $(V, s) \models \gamma^*$, and thus $(V, \mathrm{AD}(s)) \models \gamma$.

Definition 19. A complete theory T is said to have the *first isolation property* if there exist a model M of the complete theory T and an infinite set I which is an indiscernible sequence in M such that

> for any special structure (N, J) elementarily equivalent to the structure (M, I), any pseudo-finite subset A of the set J in N, any finite subset C of the model N, and any element a of N there is a countable subset $A_0 \subseteq A$ such that
>
> $$\mathrm{tp}(a/(A_0 \cup C))$$
>
> isolates $\mathrm{tp}(a/(A \cup C))$ in N.

In this case we also say that (M, I) has the first isolation property.

Definition 20. A complete theory T is said to have the *second isolation property* if there exist a model M of the complete theory T and an infinite set I which is an indiscernible sequence in (M, I) such that

> for any special structure (N, J) elementarily equivalent to the structure (M, I), any pseudo-finite subset A of the set J in N, any finite subset C of the model N, and any element a of N there is a countable subset $A_0 \subseteq A$ such that
>
> $$\mathrm{tp}(a/(A_0 \cup C))$$
>
> isolates $\mathrm{tp}(a/(A \cup C))$ in (N, J).

In this case we also say that (M, I) has the second isolation property.

Theorem 6.4. *If the theory T has the first (second) isolation property, then it has the first (second) pseudo-finite homogeneity property.*

Proof. Let M be a model of the theory T. Let $(N, J) \equiv (M, I)$, let A and B be pseudo-finite subsets of the set J in the model N, let C and D be finite subsets of N, let $h\colon (A \cup C) \to (B \cup D)$ be an elementary map in N (in (N, J)) with ω-saturated tuple (N, A, B, h) ((N, J, A, B, h), respectively), and let $a \in N$.

We find an element $b \in N$ such that $h \cup \{(a, b)\}$ is an elementary map in N (in (N, J), respectively).

We consider a special model (N_1, J_1) which is an elementary extension of (N, J) with uncountable cardinality $\kappa = \kappa^*$ such that $\mathrm{cf}(\kappa) > \omega$. There is a countable subset $A_0 \subseteq A$ such that

$$p_0 = \mathrm{tp}(a/(A_0 \cup C))$$

isolates $p = \mathrm{tp}(a/(A \cup C))$ in N_1 (in (N_1, J_1), respectively). Since h is an elementary map, it follows that $h(p)$ is a type over $(B \cup D)$ and $h(p_0)$ isolates $h(p)$. As usual, for a set q of formulae of signature $L(A \cup C)$ containing no free variables distinct from x we denote by $h(q)$ the set $\{\theta(x, h(\overline{c})) \mid \theta(x, \overline{c}) \in q\}$ of formulae. In other words, in every formula we replace every constant $c \in (A \cup C)$ by the constant $h(c)$. Since $\mathrm{cf}(\kappa) > \omega$, it follows that the system (N_1, J_1) is ω^+-saturated. Hence, there is a $b_1 \in N_1$ realizing $h(p_0)$, and hence $h(p)$ as well. Therefore, $h \cup \{(a, b_1)\}$ is an elementary map in N_1 (in (N_1, J_1), respectively). However, since (N, A, B, h) ((N, J, A, B, h)) is ω-saturated, there is a $b \in N$ such that $h \cup \{(a, b)\}$ is an elementary map in N (in (N, J), respectively). This proves Theorem 6.4.

Formulae of signature $\{<\}$ are called *order formulae.*

Definition 21. The expression $(\exists\, x \in P)\Psi$ is an abbreviation for

$$(\exists\, x)(P(x) \wedge \Psi),$$

and the expression $(\forall\, x \in P)\Psi$ is an abbreviation for

$$(\forall\, x)(P(x) \to \Psi).$$

Let K be a family of (L, P)-formulae. An (L, P)-structure (M, I) is said to be (P, K)-*reducible* if

for any formula $\phi(\overline{x}, \overline{y})$ in K there is a *reducing* quantifier-free order formula $\psi(\overline{w}, \overline{y})$ such that for any sequence $\overline{m}$ of elements of M there is a sequence $\overline{c}_{\overline{m}} \in I$ for which

$$(\forall\, \overline{y} \in P)(\psi(\overline{c}_{\overline{m}}, \overline{y}) \leftrightarrow \phi(\overline{m}, \overline{y})).$$

An (L, P)-structure (M, I) is said to be *effectively* (P, K)-*reducible* if there is an algorithm that

constructs for any formula $\phi(\overline{x}, \overline{y})$ in K a *reducing* quantifier-free order formula $\psi(\overline{w}, \overline{y})$ such that for any sequence $\overline{m}$ of elements in M there is a sequence $\overline{c}_{\overline{m}} \in I$ for which

$$(\forall\, \overline{y} \in P)(\psi(\overline{c}_{\overline{m}}, \overline{y}) \leftrightarrow \phi(\overline{m}, \overline{y})).$$

If K is the set of all L-formulae, then (P, K)-reducibility is called *P-reducibility.* If K consists of all (L, P)-formulae, then in what follows (P, K)-reducibility is called *strong P-reducibility.*

Definition 22. An (L, P)-formula is said to be *P-bounded* if it does not contain P or is of the form $(\forall\, x \in P)\Psi$ or $(\exists\, x \in P)\Psi$, where Ψ is a P-bounded formula. The quantifier $(\forall\, x \in P)$ is called a *bounded universal quantifier* and the quantifier $(\exists\, x \in P)$ is called a *bounded existential quantifier.*

An (L, P)-structure (M, I) is said to be *P-bounded* if every (L, P)-formula is equivalent in (M, I) to some P-bounded formula.

Remark 6.5. Let $(I, <)$ be a dense linear order without end elements. Every order formula $\psi(\overline{y})$ is equivalent on $(I, <)$ to a quantifier-free order formula.

Proof. The formula $(\exists\, x)(y < x < z)$ is equivalent to the formula $y < z$.

Lemma 6.6 (see [5] and [15], Theorem 2.5). *Every P-reducible and P-bounded (L, P)-structure (M, I) in which I is densely ordered without end elements is strongly P-reducible.*

Proof. Suppose that for a formula $\phi(\overline{x}, \overline{y}, z)$ there is a quantifier-free order formula $\psi(\overline{w}, \overline{y}, z)$ such that for any sequence $\overline{m}$ there is a sequence $\overline{c}_{\overline{m}} \in I$ for which

$$(\forall\, \overline{y} \in P)(\forall\, z \in P)(\psi(\overline{c}_{\overline{m}}, \overline{y}, z) \leftrightarrow \phi(\overline{m}, \overline{y}, z)).$$

Then

$$(\forall\, \overline{y} \in P)((\forall\, z \in P)\psi(\overline{c}_{\overline{m}}, \overline{y}, z) \leftrightarrow (\forall\, z \in P)\phi(\overline{m}, \overline{y}, z)).$$

Since I is densely ordered without end elements, it follows that the formula

$$(\forall\, z \in P)\psi(\overline{w}, \overline{y}, z)$$

is equivalent in $(I, <)$ to a quantifier-free formula.

Theorem 6.7. *Let M be a model of a complete L-theory T and suppose that the infinite set I is a densely ordered indiscernible sequence in M without end points.*

If (M, I) is P-reducible and P-bounded, then I is an indiscernible sequence in (M, I) and T has the second isolation property.

Proof. We note first that I is an indiscernible sequence in (M, I).

Indeed, the set I is discerned neither by quantifier-free (L, P)-formulae nor by L-formulae. Suppose that I is not discerned by P-bounded formulae with number of P-quantifiers at most n, and consider a formula $(\exists\, x \in P)\Psi$. It is clear that if Ψ holds on the first tuple for some value of x, then Ψ holds on the second tuple ordered in the same way for a value of x whose relative location with respect to the second tuple is analogous to the location of the first value of x with respect to the first tuple. The consideration of the formula $(\forall\, x \in P)\Psi$ reduces to the remark that for any value of x there is another value of x whose location with respect to the first tuple is analogous to the location of the first value of x with respect to the second tuple.

Let us now prove the isolation property. We consider an arbitrary pseudo-finite set $A \subseteq I$ and a finite set C. We take an arbitrary element $a \in M$ and a finite sequence $\overline{m}$ of elements in C. For any (L,P)-formula $\phi(z,\overline{x},\overline{y})$ there is a quantifier-free order formula $\psi_\phi(\overline{w},\overline{y})$ such that

$$(\forall\,\overline{y}\in P)(\psi_\phi(\overline{c}_{\overline{m},a},\overline{y})\leftrightarrow\phi(a,\overline{m},\overline{y})) \tag{4}$$

for some sequence $\overline{c}_{\overline{m},a}\in I$. For any $\phi(z,\overline{x},\overline{y})$ we fix a sequence $\overline{c}_{\overline{m},a}$ for which the formula (4) holds. Since A is pseudo-finite, it follows that one of the following four possibilities holds for any element b of the sequence $\overline{c}_{\overline{m},a}$: it belongs to A, and we include b in A_ϕ; it is greater than all elements in A, and we include the greatest element of A in A_ϕ; it is less than any element of A, and we include the least element of A in A_ϕ; there exist a greatest element ${}_ba$ in A among the elements less than b and a least element a_b in A among the elements greater than b, and we include both the extreme elements ${}_ba$ and a_b in A_ϕ. It is clear that the order quantifier-free type of the sequence $\overline{d}\in A$ over $\overline{c}_{\overline{m},a}$ is determined by the order quantifier-free type of this sequence $\overline{d}\in A$ over A_ϕ. The union A_0 of all the sets A_ϕ for all ϕ is countable. It is clear that the type of a over A_0 in $(M,I,\overline{m})$ isolates the type of a over A in $(M,I,\overline{m})$.

7. Reducible theories

Definition 23 (reducible theories). A universe U of finite signature L is said to be *reducible* if there is a P-reducible (L,P)-structure (M,I) such that $M\equiv U$ and I is an infinite indiscernible sequence in M. For a reducible universe U the theory $\mathrm{Th}(U)$ is also said to be *reducible.* A reducible universe U and its theory are said to be *effectively reducible* if there is an effectively P-reducible (L,P)-structure (M,I) such that $M\equiv U$ and I is an infinite indiscernible sequence in M.

Theorem 7.1. *For any reducible universe U of finite signature L there is a P-reducible (L,P)-structure (N,J) such that $N\equiv U$, J is an indiscernible sequence in N for which the restriction of $<$ to J defines a linear order on J such that $(J,<)$ is the set of reals with the standard ordering, and N is a $(2^\omega)^+$-saturated system.*

Proof. We consider an arbitrary P-reducible (L,P)-structure (M,I) such that $M\equiv U$, and let I be an infinite indiscernible sequence in M. Let us choose an uncountable cardinal κ greater than the cardinality of M and such that $\mathrm{cf}(\kappa)>2^\omega$ and $\kappa=\kappa^*$. We consider a special elementary extension (V,J) of cardinality κ of the (L,P)-structure (M,I). It follows from the condition $\mathrm{cf}(\kappa)>2^\omega$ that (V,J) is a $(2^\omega)^+$-saturated system. By the definition of $(2^\omega)^+$-saturation, we can regard $(J,<)$ as an extension of the set $\mathbb{R}$ of reals ordered in the standard way. Since $(V,J)\succeq(M,I)$, (V,J) is also a P-reducible (L,P)-structure.

It remains to prove that $(V,\mathbb{R})$ is a P-reducible (L,P)-structure.

Consider an arbitrary L-formula $\phi(\overline{x},\overline{y})$. For this formula $\phi(\overline{x},\overline{y})$ there is a quantifier-free order formula $\psi(\overline{w},\overline{y})$ such that for any sequence $\overline{m}$ of elements in V there is a sequence $\overline{c}_{\overline{m}}\in J$ for which the formula

$$(\forall\,\overline{y}\in P)(\psi(\overline{c}_{\overline{m}},\overline{y})\leftrightarrow\phi(\overline{m},\overline{y}))$$

holds in (V,J). We can assume that $\psi(\overline{w},\overline{y})$ is a disjunction of conjunctions of formulae of the form $u<v$ and $u=v$ in which u and v are elements of the tuples $\overline{w}$ and $\overline{y}$.

Our objective is to find an order formula

$$\theta(\overline{w}, \overline{u}, \overline{z}, \overline{y})$$

such that for any sequence $\overline{m}$ of elements in V there are sequences $\overline{a}_{\overline{m}} \in \mathbb{R}$, $\overline{d}_{\overline{m}} \in \mathbb{R}$, and $\overline{e}_{\overline{m}} \in \mathbb{R}$ for which the formula

$$(\forall \overline{y} \in P)(\theta(\overline{a}_{\overline{m}}, \overline{d}_{\overline{m}}, \overline{e}_{\overline{m}}, \overline{y}) \leftrightarrow \phi(\overline{m}, \overline{y}))$$

holds in $(V, \mathbb{R})$.

The tuple $\overline{u}$ of variables is the tuple of doubles for the variables in the tuple $\overline{w}$. The tuples $\overline{w}$, $\overline{u}$, and $\overline{y}$ do not contain variables in common. We shall define the tuple $\overline{z}$ of variables below.

Let us take an arbitrary sequence $\overline{m}$ of elements in V and the sequence $\overline{c}_{\overline{m}} \in J$.

Consider an arbitrary order-preserving map of the elements of the sequence $\overline{c}_{\overline{m}}$ into $\mathbb{R}$. Let this map take the sequence $\overline{c}_{\overline{m}}$ to a sequence $\overline{a}_{\overline{m}} \in \mathbb{R}$. We thus choose the values $\overline{a}_{\overline{m}}$ for the variables $\overline{w}$.

We consider an arbitrary element c in $\overline{c}_{\overline{m}}$ which is the value of the variable w_i, and we find the value $d(c)$ for the double u_i of this variable.

If $c \in \mathbb{R}$, then we set $d(c) = c$. If c is greater than all the elements of $\mathbb{R}$, then we set $d(c) = \infty$. If c is less than all the elements of $\mathbb{R}$, then we set $d(c) = -\infty$.

If none of the above cases holds, then c partitions the reals into a non-empty class of real numbers less than c and a non-empty class of real numbers greater than c. In this case we take $d(c)$ to be either the greatest number in the first class or the least number in the second class.

We use the sequence $d(\overline{c}_{\overline{m}})$ as the sequence of values for the sequence $\overline{u}$ of variables.

Let us correct the formula $\psi(\overline{w}, \overline{y})$ as follows. In every equality and inequality containing a variable in the tuple $\overline{y}$ we replace every variable in the tuple $\overline{w}$ by the double of this variable in the tuple $\overline{u}$. We obtain some formula

$$\theta^*(\overline{w}, \overline{u}, \overline{y}).$$

The value c of the variable w_i in $\overline{w}$ can be in $\mathbb{R}$, or can exceed all elements of $\mathbb{R}$, or can be less than any element of $\mathbb{R}$, or can split the reals into the non-empty class of real numbers less than c and the non-empty class of real numbers greater than c. In the last case, either there is a greatest number in the first class or there is a least number in the second class.

For any variable w_i in $\overline{w}$ that occurs in $\psi(\overline{w}, \overline{y})$ in equalities and inequalities with variables in $\overline{y}$ we consider all these five cases. If there are k such variables, then we consider 5^k cases in all.

We correct the equalities and inequalities with variables in $\overline{y}$ in each of the 5^k cases in the formula

$$\theta^*(\overline{w}, \overline{u}, \overline{y})$$

as follows.

If the value c of the variable w_i in $\overline{w}$ belongs to $\mathbb{R}$, then the equalities and inequalities with u_i are not corrected.

If the value c of the variable w_i in $\overline{w}$ is greater than any element of $\mathbb{R}$, then we regard the inequalities $y_j < u_i$ as true and replace them by $y_j = y_j$, and we regard the inequalities $u_i < y_j$ and the equalities $y_j = u_i$ and $u_i = y_j$ as false and replace them by $y_j < y_j$.

If the value c of the variable w_i in $\overline{w}$ is less than any element of $\mathbb{R}$, then we regard the inequalities $u_i < y_j$ as true and replace them by $y_j = y_j$, and we regard the inequalities $y_j < u_i$ and the equalities $u_i = y_j$ and $y_j = u_i$ as false and replace them by $y_j < y_j$.

If the value c of the variable w_i in $\overline{w}$ is not in $\mathbb{R}$ and partitions the reals into the non-empty class of real numbers less than c and the non-empty class of real numbers greater than c, and if the first class contains a largest number, then we replace every inequality $y_j < u_i$ by

$$(y_j < u_i \vee y_j = u_i).$$

In this case we regard the equalities $y_j = u_i$ and $u_i = y_j$ as false and replace them by $y_j < y_j$.

If the value c of the variable w_i in $\overline{w}$ is not in $\mathbb{R}$ and partitions the reals into the non-empty class of real numbers less than c and the non-empty class of real numbers greater than c, and if the second class contains a least number, then we replace every inequality $u_i < y_j$ by

$$(u_i < y_j \vee y_j = u_i).$$

In this case we regard the equalities $y_j = u_i$ and $u_i = y_j$ as false and replace them by $y_j < y_j$.

For the ith case we obtain the formula $\theta_i(\overline{w}, \overline{u}, \overline{y})$.

Let $z_0, z_1, \ldots, z_{5^k}$ be pairwise distinct variables not appearing in any of the tuples $\overline{u}$, $\overline{y}$, and $\overline{w}$. Let

$$\overline{z} = z_0, z_1, \ldots, z_{5^k}.$$

It is clear that for

$$\theta(\overline{w}, \overline{u}, \overline{z}, \overline{y})$$

one can take

$$\left(z_1 < \cdots < z_{5^k} \wedge \bigvee_{i=1}^{5^k} (\theta_i(\overline{w}, \overline{u}, \overline{y}) \wedge z_0 = z_i)\right).$$

Definition 24 (independent formula [19], [20], [5]). Let M be an L-structure. An L-formula $\phi(\overline{x}, \overline{y})$ is said to be an *independent formula* in M if for any natural number n there are tuples

$$\overline{a}_1, \ldots, \overline{a}_n$$

of values for the tuple $\overline{x}$ of variables such that the following condition holds:

(A) for any $\eta \subseteq \{1, \ldots, n\}$ there is a tuple $\overline{b}_\eta$ of values for the tuple $\overline{y}$ of variables such that

$$\eta = \{i \in \{1, \ldots, n\} \mid M \models \phi(\overline{a}_i, \overline{b}_\eta)\}.$$

Theorem 7.2 ([5], Theorem 5.2). *Let an infinite set I be an indiscernible densely completely ordered sequence without end elements in the universe M of signature L.*

If M admits no independent formula, then (M, I) is P-reducible.

Proof. Consider an arbitrary L-formula $\phi(\overline{x}, \overline{y})$.

Since $\phi(\overline{x}, \overline{y})$ is not an independent formula in M, there is an n for which any tuples

$$\overline{a}_1, \ldots, \overline{a}_n$$

of values for a tuple $\overline{x}$ of variables in the support of the structure M, and, all the more so, in I, do not satisfy the condition (A).

Let $\overline{x} = x_1, \ldots, x_m$. Let $\overline{x}_i = x_{i,1}, \ldots, x_{i,m}$ for $i = 1, 2$.

By an order quantifier-free type $p(\overline{x}_1, \overline{x}_2)$ of the variables $(\overline{x}_1, \overline{x}_2)$ over the empty set we mean a satisfiable set of formulae of the forms $x_{i,j} < x_{k,l}$ and $x_{i,j} = x_{k,l}$ with $i, k \in \{1, 2\}$ and $j, l \in \{1, \ldots, m\}$ which is not contained in any other satisfiable set of such formulae. A type $p(\overline{x}_1, \overline{x}_2)$ is said to be realized by a tuple $(\overline{a}_i, \overline{a}_j)$ of elements of the set I if all the formulae in $p(\overline{a}_i, \overline{a}_j)$ are true, in other words, if all formulae in $p(\overline{x}_1, \overline{x}_2)$ become true after replacing $x_{1,k}$ by $a_{i,k}$ and $x_{2,k}$ by $a_{j,k}$.

It is clear that for a given natural positive number m one can find a natural positive number $\tau(m)$ such that there are exactly $\tau(m)$ different order quantifier-free types of a fixed $2m$-tuple of variables over the empty set. Therefore, by the Ramsey theorem on finite sets, any sequence

$$\overline{a}_1, \ldots, \overline{a}_{r(2,\tau(m),n+1)}$$

contains a subsequence of length $n+1$ in which every pair of terms with first element preceding the second in the sequence realizes the same order quantifier-free type over the empty set. Let $r = r(2, \tau(m), n + 1)$.

By the order quantifier-free type $p(\overline{x}, C)$ of the variables $\overline{x} = x_1, \ldots, x_m$ over a given finite set $C \subseteq I$ we mean a satisfiable set of formulae of the forms $x_j < x_l$, $x_j = x_l$, $x_j < c$, $c < x_j$, and $x_j = c$ with $j, l \in \{1, \ldots, m\}$ and $c \in C$ which is not contained in any other satisfiable set of such formulae. A type $p(\overline{x}, C)$ is said to be realized by a tuple $\overline{a}$ of elements of the set I if all the formulae in $p(\overline{a}, C)$ are true, in other words, if all formulae in $p(\overline{x}, C)$ become true after replacing x_j by a_j for $j = 1, \ldots, n$.

Let $c_1, d_1, \ldots, c_m, d_m$ be elements of the set I.

By a neighbourhood $(c_1, d_1), \ldots, (c_m, d_m)$ of a tuple $\overline{a}$ in I^m such that $c_1 < a_1 < d_1$, $\ldots$, $c_m < a_m < d_m$, we mean the set of all tuples $\overline{b}$ of elements of I such that $c_1 < b_1 < d_1$, $\ldots$, $c_m < b_m < d_m$.

By a neighbourhood of a tuple $\overline{a}$ in a type $p(\overline{x}, C)$ realized by a tuple $\overline{a}$ we mean the intersection of a neighbourhood of $\overline{a}$ in I^m with the set of all tuples realizing the type $p(\overline{x}, C)$.

Let us fix a tuple $\overline{b}$ of elements of the support of M with the same length as the tuple $\overline{y}$. Let

$$V_{\overline{b}} = \{\overline{d} \in I \mid M \models \phi(\overline{d}, \overline{b})\}.$$

A sequence

$$\overline{a}_1, \ldots, \overline{a}_i$$

of tuples of elements of the set I is said to be *coherent* if for any $j \in \{1, \dots, i\}$ any neighbourhood of $\overline{a}_j$ in a type $p(\overline{x}, C_j)$ realized by $\overline{a}_j$ contains both tuples belonging to $V_{\overline{b}}$ and tuples not belonging to $V_{\overline{b}}$. Here C_j stands for the set of all elements appearing in at least one tuple in the sequence

$$\overline{a}_1, \dots, \overline{a}_{j-1}.$$

For $j = 1$ the set C_j is empty.

A coherent sequence

$$\overline{a}_1, \dots, \overline{a}_i$$

is said to be a *covering* if it cannot be extended to a longer coherent sequence.

Lemma 7.3. *There is no coherent sequence of length r.*

Proof of the lemma. In a coherent sequence of length r one can find a subsequence

$$\overline{d}_0, \overline{d}_1, \dots, \overline{d}_n$$

of length $n+1$ such that every pair of terms with first element preceding the seecond in the sequence realizes the same order quantifier-free type over the empty set.

We preserve the tuple $\overline{d}_0$. Let us correct the other tuples as follows. If in this subsequence some element of a subsequent tuple is equal to some element of a preceding tuple, then there is a place such that the elements at this place in any two tuples in the subsequence are the same. In this case we delete this place from every tuple of the subsequence. After this, every element of a subsequent tuple differs from every element of a preceding tuple. If in some tuple of the subsequence there are equal elements at two distinct places, then these places are filled by equal elements in every tuple of the subsequence. We arrange the elements of the tuple $\overline{d}_0$ and the remaining elements of all other tuples in the subsequence in ascending order. For the elements u_i and u_{i-1} located at the ith and $(i-1)$th places we choose $v_i \in I$ so that $u_{i-1} < v_i < u_i$. We also assume that $v_1 < u_1$ and $u_k < v_{k+1}$ for the greatest remaining element u_k. We consider the neighbourhoods (v_i, v_{i+1}) of the elements u_i. By a neighbourhood of a tuple in the subsequence we mean a tuple formed by the deleted elements of the tuple in question and neighbourhoods of the remaining elements of the tuple.

Consider an arbitrary $\eta \subseteq \{1, \dots, n\}$. If $i \in \eta$, then in the neighbourhood under consideration of the tuple $\overline{d}_i$ we choose a tuple $\overline{c}_i$ belonging to $V_{\overline{b}}$. If $i \notin \eta$, then in the neighbourhood under consideration (of the tuple $\overline{d}_i$) we choose a tuple $\overline{c}_i$ not belonging to $V_{\overline{b}}$. By construction, the sequences

$$\overline{d}_1, \dots, \overline{d}_n$$

and

$$\overline{c}_1, \dots, \overline{c}_n$$

are equally ordered. Since I is indiscernible, there is a tuple $\overline{b}_\eta$ such that

$$\eta = \{i \in \{1, \dots, n\} \mid M \models \phi(\overline{d}_i, \overline{b}_\eta)\}.$$

However, this contradicts the choice of n, and thus proves Lemma 7.3.

It follows from the lemma that every coherent sequence can be extended to a covering. Let

$$\overline{a}_1, \dots, \overline{a}_i$$

be a covering. This means that for any $\overline{a}_{i+1} \in I$ there is a neighbourhood of $\overline{a}_{i+1}$ in a type $p(\overline{x}, C_{i+1})$ realized by $\overline{a}_{i+1}$ which contains either only tuples belonging to $V_{\overline{b}}$ or only tuples not belonging to $V_{\overline{b}}$.

Lemma 7.4. *Let $D \subseteq I$ satisfy the following condition:*

for any $\overline{a}_{i+1} \in I$ there is a neighbourhood of $\overline{a}_{i+1}$ in a type $p(\overline{x}, D)$ realized by $\overline{a}_{i+1}$ such that this neighbourhood contains either only tuples of the first kind or only tuples of the second kind.

For any type $p(\overline{x}, D)$ either all tuples $\overline{a}_{i+1}$ realizing this type are of the first kind or all tuples $\overline{a}_{i+1}$ realizing this type are of the second kind.

Proof of the lemma. We shall say for brevity that a tuple belongs to a type if it realizes the type. We argue by induction on m. For $m = 1$ the type $p(x, D)$ is given by the condition $x = c$, or by the condition $x < c$, or by the condition $c < x$, or by the condition $c < x < d$. We consider only the last case, because the statement obviously holds in the first case and the proof is similar in the remaining two cases. Suppose that the type in question contains both elements of the first kind and elements of the second kind. We choose in this type an arbitrary element a of the first kind and an arbitrary element e of the second kind. To be definite, suppose that $a < e$. The other case can be treated similarly. Let us refer an element a' in this type to the first part if a' does not exceed a or if the whole interval (a, a') contains only elements of the first kind. We refer the other elements of this type to the second part. It is clear that every element of the first part is less than every element of the second part. Both the parts are non-empty. Since the order on I is complete, it follows that either the first part contains a largest element or the second part has a least element. If a_1 is such an element, then we consider a neighbourhood of a_1 in which either all elements are of the first kind or all elements are of the second kind. The second case is impossible, whereas the first case contradicts the choice of a_1.

Suppose that the statement has been proved for $m - 1$. To any type $p(\overline{x}, D)$ we assign the type

$$p'(x_1, \dots, x_{m-1}, D),$$

obtained by removing the formulae containing x_m. For any tuple $a_1, \dots, a_{m-1}$ realizing the type $p'(x_1, \dots, x_{m-1}, D)$ we denote by A the set of all elements of the tuple $a_1, \dots, a_{m-1}$. Let

$$p''(x_m, D \cup A)$$

denote the type obtained from $p(\overline{x}, D)$ by replacing $x_1, \dots, x_{m-1}$ by $a_1, \dots, a_{m-1}$, respectively. One can easily see that every element a_m in $p''(x_m, D \cup A)$ has a neighbourhood all of whose elements, together with $a_1, \dots, a_{m-1}$, are either of the first kind or of the second kind. It follows from what was said above that either all elements of $p''(x_m, D \cup A)$ together with $a_1, \dots, a_{m-1}$ are of the first kind or all are of the second kind. We regard $a_1, \dots, a_{m-1}$ as being of the first kind if all the elements in $p''(x_m, D \cup A)$, together with $a_1, \dots, a_{m-1}$, are of the first kind and as being of the second kind if all the elements in $p''(x_m, D \cup A)$, together with $a_1, \dots, a_{m-1}$, are of the second kind. Every tuple $a_1, \dots, a_{m-1}$ in

$$p'(x_1, \dots, x_{m-1}, D)$$

has a neighbourhood in which all tuples are of the same kind. By induction, all tuples in $p'(x_1, \dots, x_{m-1}, D)$ are of the same kind. This proves Lemma 7.4.

Thus, every type $p(\overline{x}, C_{i+1})$ consists either only of tuples belonging to $V_{\overline{b}}$ or only of tuples not belonging to $V_{\overline{b}}$. This means that $V_{\overline{b}}$ is a union of several types.

Let C_{i+1} contain k elements. To each type we assign the conjunction of all formulae occurring in this type, where the elements in C_{i+1} in these formulae are replaced by the variables $z_1, \dots, z_k$, and we consider all possible disjunctions of the formulae thus obtained. Let there be l disjunctions $\Psi_1, \dots, \Psi_l$. Finally, let a tuple $\overline{z}$ contain $k + l + 1$ variables. As a reducing formula for $\phi(\overline{x}, \overline{y})$ one can use the order formula

$$\bigvee_{j=1}^{l} (z_{k+l+1} = z_{k+j} \wedge \Psi_j).$$

This proves Theorem 7.2.

In conclusion we present another result showing that the reducibility property and the relative isolation property are equivalent in a certain sense.

We say that tuples $\overline{d}_1 = (d_1^1, \dots, d_1^m)$ and $\overline{d}_2 = (d_2^1, \dots, d_2^m)$ with the same length as $\overline{x}$ in an indiscernible sequence I in an L-structure M are a discerning pair for an L-formula $\phi(\overline{x}, \overline{y})$ and a tuple $\overline{a}$ of elements in M with the same length as the tuple $\overline{y}$ if:

1) the elements of these tuples are equally ordered, in other words, for any $1 \leqslant i, j \leqslant m$ the relation $d_1^i < d_1^j$ holds if and only if $d_2^i < d_2^j$;
2) there is an i_0 such that $d_1^i = d_2^i$ for any i in $\{1, \dots, m\}$ different from i_0;
3) the truth values of $\phi(\overline{d}_1, \overline{a})$ and $\phi(\overline{d}_2, \overline{a})$ in M are different.

In this case, i_0 is said to be a non-equal coordinate of the discerning pair in question. If $d_1^{i_0} < d_2^{i_0}$, then the segment

$$[d_1^{i_0}, d_2^{i_0}] = \{a \in I \mid d_1^{i_0} \leqslant a \leqslant d_2^{i_0}\}$$

is said to be discerning for the pair $(\overline{d}_1, \overline{d}_2)$ and is denoted by $\mathrm{diff}[\overline{d}_1, \overline{d}_2]$.

An element e of the set I is said to be defining for an L-formula $\phi(\overline{x}, \overline{y})$ and a tuple $\overline{a}$ of elements in M if for any neighbourhood O in I of the element e there is a discerning pair $\overline{d}_1$, $\overline{d}_2$ for $\phi(\overline{x}, \overline{y})$ and $\overline{a}$ such that $\mathrm{diff}[\overline{d}_1, \overline{d}_2]$ is contained in O.

For brevity, we say that an L-formula $\phi(\overline{x}, \overline{y})$ is *P-reducible* in an (L, P)-structure (M, I) if there is a reducing quantifier-free order formula for the formula $\phi(\overline{x}, \overline{y})$.

It is quite clear that an L-formula $\phi(\overline{x}, \overline{y})$ is P-reducible in an (L, P)-structure (M, I) in which I is a densely completely ordered indiscernible sequence if and only if there is a natural number k such that for any tuple $\overline{a}$ in M the number of defining elements for ϕ and $\overline{a}$ does not exceed k.

Theorem 7.5. *If an (L, P)-structure (M, I) has the first isolation property, then there exist a pair $(N, J') \equiv (M, I)$ and an infinite subset J of the set J' such that (N, J) is P-reducible.*

Proof. Consider a $(2^\omega)^+$-saturated structure (N, J') elementarily equivalent to the structure (M, I). It is clear that there is a subset J of J' such that $(J, <)$ is isomorphic to the reals ordered in the standard way.

Suppose that the structure (N, J) is not P-reducible. Let there be no reducing formula for an L-formula $\phi(\overline{x}, \overline{y}, z)$ and let a reducing formula exist for any L-formula $\theta(\overline{v}, \overline{u})$ for which the length of the tuple $\overline{u}$ is less than the length of the tuple $\overline{y}, z$.

Since the L-formula $\phi(\overline{x}, \overline{y}, z)$ is not P-reducible, it follows that for any positive integer i there is a tuple $(\overline{a}_i, b_i)$ for which the number of elements for the L-formula $\phi(\overline{x}, \overline{y}, z)$ and the tuple $(\overline{a}_i, b_i)$ exceeds i. Let us choose pairwise distinct elements $e_{i1}, \dots, e_{ii}$ among the defining elements for the L-formula $\phi(\overline{x}, \overline{y}, z)$ and the tuple $(\overline{a}_i, b_i)$. For any chosen defining element e_{ij} we choose a discerning pair $(\overline{d}_1^{ij}, \overline{d}_2^{ij})$ in such a way that the segments $\mathrm{diff}[\overline{d}_1^{ij}, \overline{d}_2^{ij}]$ are disjoint for distinct indices j. The triple $(\overline{d}_1^{ij}, \overline{d}_2^{ij}, \overline{d}_3^{ij})$ is defined by the rule that the tuple $\overline{d}_3^{ij}$ is obtained from the tuple $\overline{d}_2^{ij}$ when the element at the place of the non-equal coordinate of this discerning pair is moved to the interior of the segment $\mathrm{diff}[\overline{d}_1^{ij}, \overline{d}_2^{ij}]$. It is clear that one of the pairs $(\overline{d}_1^{ij}, \overline{d}_3^{ij})$ and $(\overline{d}_3^{ij}, \overline{d}_2^{ij})$ is not discerning.

It is clear that there is an ω^+-saturated elementary extension (N', J''', J'') of the structure (N, J', J) such that one can find a tuple $(\overline{a}, b)$ in N' for which there are ω^+ defining elements e_j satisfying the condition that the set D of all elements of the triples $(\overline{d}_1^j, \overline{d}_2^j, \overline{d}_3^j)$ constructed as described above is a pseudo-finite set in N' of cardinality at least ω^+. It is clear that $D \subseteq J''$ and that the structure (N', J''') is elementarily equivalent to the structure (M, I). Let E be the set of all elements of the tuple $\overline{a}$.

Let us consider the type $p = \mathrm{tp}(b/(E \cup D))$ and assume that p is isolated by the subtype $p_0 = \mathrm{tp}(b/(E \cup D_0))$ for some countable subset D_0 of D. We assume that z is the only free variable in the formulae in p.

It is clear that there are ω^+ pairs $(\overline{d}_1^j, \overline{d}_2^j)$ neither of whose elements belongs to D_0 and for which the segments $\mathrm{diff}[\overline{d}_1^j, \overline{d}_2^j]$ contain no elements of the set D_0. These pairs are said to be *distinguished.*

Let a formula $\Phi(\overline{u}, \overline{v}, \overline{y}, z)$ say that $(\overline{u}, \overline{v})$ is a discerning pair for the L-formula $\phi(\overline{x}, \overline{y}, z)$ and the tuple $(\overline{y}, z)$. It is clear that $p_0 \cup \{\Phi(\overline{d}_1, \overline{d}_2, \overline{a}, z)\}$ is satisfiable for any $(\overline{d}_1, \overline{d}_2)$. We show that $p_0 \cup \{\neg\Phi(\overline{d}_1, \overline{d}_2, \overline{a}, z)\}$ is also satisfiable in N' for some distinguished pair $(\overline{d}_1, \overline{d}_2)$. If this is the case, then the type p_0 does not isolate the type p.

Suppose that the set $p_0 \cup \{\neg\Phi(\overline{d}_1, \overline{d}_2, \overline{a}, z)\}$ is not satisfiable in N' for any distinguished pair $(\overline{d}_1, \overline{d}_2)$. In this case, for any distinguished pair $(\overline{d}_1, \overline{d}_2)$ there is a finite subset $p_0(\overline{d}_1, \overline{d}_2)$ of type p_0 for which $p_0(\overline{d}_1, \overline{d}_2) \cup \{\neg\Phi(\overline{d}_1, \overline{d}_2, \overline{a}, z)\}$ is not satisfiable in N'. However, it is clear that there are at least ω^+ distinct distinguished pairs $(\overline{d}_1, \overline{d}_2)$ with the same subset $p'_0 = p_0(\overline{d}_1, \overline{d}_2)$.

Since the set $p'_0 \cup \{\neg\Phi(\overline{d}_1, \overline{d}_2, \overline{a}, z)\}$ of formulae is finite, one can consider the conjunction $\Psi(\overline{d}, \overline{d}_1, \overline{d}_2, \overline{a}, z)$ of all formulae in this set, where $\overline{d}$ in the conjunction stands for the elements of D_0. The formula $(\exists\, z)\Psi(\overline{w}, \overline{u}, \overline{v}, \overline{y}, z)$ is P-reducible in the structure (N', J'''). Let $\overline{c}$ be the defining elements for this formula and the tuple $\overline{a}$.

Let the elements of the tuples $\overline{d}_1$, $\overline{d}_2$, and $\overline{d}_3$ be equally located with respect to $\overline{c}$ and $\overline{d}$, let the pair $(\overline{d}_1, \overline{d}_2)$ be discerning for $(\overline{a}, b)$, and let the pair $(\overline{d}_1, \overline{d}_3)$ be not discerning for $(\overline{a}, b)$. We recall that the tuple $\overline{d}_3$ is obtained from the tuple $\overline{d}_2$ when the element at the place of the non-equal coordinate of the discerning pair $(\overline{d}_1, \overline{d}_2)$ is moved to the interior of the segment $\mathrm{diff}[\overline{d}_1, \overline{d}_2]$. In this case the validity of the formula $(\exists\, z)\Psi(\overline{d}, \overline{d}_1, \overline{d}_2, \overline{a}, z)$ is equivalent to the validity of the formula $(\exists\, z)\Psi(\overline{d}, \overline{d}_1, \overline{d}_3, \overline{a}, z)$. However, the formula $\Psi(\overline{d}, \overline{d}_1, \overline{d}_3, \overline{a}, z)$ is true for $z = b$. Therefore, the formula $(\exists\, z)\Psi(\overline{d}, \overline{d}_1, \overline{d}_2, \overline{a}, z)$ is true as well.

8. Boundedness of reducible theories

Definition 25. A P-reducible (L, P)-structure (N, J) such that $N \equiv U$, J is an indiscernible sequence in N, the restriction of $<$ to J defines a linear order such that $(J, <)$ is the set of the reals with the standard ordering, and N is a $(2^\omega)^+$-saturated system is called a *small model* for the universe U and for the elementary theory of this universe.

By Theorem 7.1, every reducible theory has a small model.

Definition 26. A reducible theory is said to be *bounded* if every small model of this theory is P-bounded.

Theorem 8.1. *Every reducible theory is bounded.*

This theorem and Theorem 6.7 imply the following corollary.

Corollary 8.2. *Every reducible theory has the second isolation property.*

This, together with Theorems 6.3 and 6.4, implies the following corollary.

Corollary 8.3. *For any reducible universe U every extended query locally generic for finite states over U is equivalent to some restricted query for finite states over U.*

This, together with Theorem 7.2, implies the following result.

Corollary 8.4. *For any universe U without an independent formula every extended query locally generic for finite states over U is equivalent to some restricted query for finite states over U.*

The rest of this section is devoted to the proof of Theorem 8.1. Both the theorem and its proof are due to Dudakov (see [8]).

The constructions below are based on the following remark.

Remark 8.5. Let $\psi(\overline{z}, \overline{y})$ be a quantifier-free order formula, let $a \leqslant b$, and let the tuples $\overline{c}$, $\overline{d}_1 = d_{1,1}, \dots, d_{1,m}$, $\overline{d}_2 = d_{2,1}, \dots, d_{2,m}$ satisfy the following conditions:

1) $d_{1,1} < \dots < d_{1,m}$ and $d_{2,1} < \dots < d_{2,m}$;
2) for any $i \in \{1, \dots, m\}$ and $j \in \{1, 2\}$ each of the conditions $d_{j,i} < a$ and $d_{j,i} > b$ implies that $d_{1,i} = d_{2,i}$;
3) $\psi(\overline{c}, \overline{d}_1)$ is true and $\psi(\overline{c}, \overline{d}_2)$ is false.

In this case there is an element e in the tuple $\overline{c}$ such that $a \leqslant e \leqslant b$.

Indeed, otherwise the tuples $\overline{d}_1$ and $\overline{d}_2$ realize the same order quantifier-free type over $\overline{c}$.

Let $(I, <)$ be a dense linear order without end elements. We recall Remark 6.5 that every order formula $\psi(\overline{y})$ is equivalent on $(I, <)$ to an quantifier-free order formula.

Below in this section we assume that (M, I) is a P-reducible (L, P)-structure and satisfies the following conditions:

> I is an indiscernible sequence in M, the restriction of $<$ to I defines a linear order such that $(I, <)$ is a densely and completely ordered set of cardinality λ without end elements, and M is a λ^+-saturated system.

Lemma 8.6 [5]. *Each P-bounded L-formula is equivalent in (M, I) to a P-bounded formula beginning with bounded existential quantifiers, followed by bounded universal quantifiers, followed by an L-formula.*

A formula beginning with bounded existential quantifiers, followed by bounded universal quantifiers, followed by an L-formula will be called a bounded ∃∀-formula.

Proof. Let $\psi_\phi(\overline{z}, \overline{w})$ be a reducing formula for $\phi(\overline{y}, \overline{w})$. In other words, for any sequence $\overline{m}$ of elements in M there is a sequence $\overline{c}_{\overline{m}} \in I$ such that

$$(\forall\, \overline{w} \in P)(\psi_\phi(\overline{c}_{\overline{m}}, \overline{w}) \leftrightarrow \phi(\overline{m}, \overline{w})).$$

Hence, the formula

$$(\forall\, \overline{y})(\exists\, \overline{z} \in P)(\forall\, \overline{w} \in P)(\psi_\phi(\overline{z}, \overline{w}) \leftrightarrow \phi(\overline{y}, \overline{w}))$$

is true in (M, I).

Let $Q_1, \ldots, Q_n$ be some symbols of quantifiers.

Then the formula

$$(Q_1 w_1 \in P) \ldots (Q_n w_n \in P)\phi(\overline{y}, \overline{w})$$

is equivalent to the formula

$$(\exists\, \overline{z} \in P)((\forall\, \overline{w} \in P)(\psi_\phi(\overline{z}, \overline{w}) \leftrightarrow \phi(\overline{y}, \overline{w})) \wedge (Q_1 w_1 \in P) \ldots (Q_n w_n \in P)\psi_\phi(\overline{z}, \overline{w})).$$

The formula

$$(Q_1 w_1 \in P) \ldots (Q_n w_n \in P)\psi_\phi(\overline{z}, \overline{w})$$

can be replaced (by Remark 6.5) by a quantifier-free formula. This proves Lemma 8.6.

To complete the proof of Theorem 8.1, it suffices to prove that a formula obtained by existential quantification of a bounded ∃∀-formula is equivalent to a P-bounded formula in (M, I).

Thus, let us consider an (L, P)-formula of the form

$$(\exists\, \overline{z} \in P)(\forall\, \overline{w} \in P)\phi(x, \overline{y}, \overline{z}, \overline{w}),$$

in which $\phi(x, \overline{y}, \overline{z}, \overline{w})$ is an L-formula. It is clear that the formula

$$(\exists\, x)(\exists\, \overline{z} \in P)(\forall\, \overline{w} \in P)\phi(x, \overline{y}, \overline{z}, \overline{w})$$

is equivalent to the formula

$$(\exists\, \overline{z} \in P)(\exists\, x)(\forall\, \overline{w} \in P)\phi(x, \overline{y}, \overline{z}, \overline{w}),$$

since the neighbouring existential quantifiers can be interchanged. For this reason, it suffices to construct a P-bounded formula equivalent in (M, I) to the formula

$$(\exists\, x)(\forall\, \overline{w} \in P)\phi(x, \overline{y}, \overline{z}, \overline{w}).$$

Since one can now combine the tuples $\overline{y}$ and $\overline{z}$, it suffices to construct a P-bounded formula equivalent in (M, I) to the formula

$$(\exists\, x)(\forall\, \overline{w} \in P)\phi(x, \overline{y}, \overline{w}),$$

in which $\phi(x, \overline{y}, \overline{w})$ is an L-formula.

Since M is λ^+-saturated, the following remark holds.

Remark 8.7. The validity of the formula

$$(\exists\, x)(\forall\, \overline{w} \in P)\phi(x, \overline{y}, \overline{w})$$

on a given tuple $\overline{b}$ is equivalent to the condition that for any finite set $S \subseteq I$ there is an a such that $\phi(a, \overline{b}, \overline{w})$ holds in M for any tuples $\overline{w}$ whose elements are taken from S.

Let us fix a quantifier-free order formula $\psi_\phi(\overline{z}, \overline{w})$ for which the formula

$$(\forall\, x)(\forall\, \overline{y})(\exists\, \overline{z} \in P)(\forall\, \overline{w} \in P)(\psi_\phi(\overline{z}, \overline{w}) \leftrightarrow \phi(x, \overline{y}, \overline{w}))$$

holds true in (M, I). Let the length of the tuple $\overline{z}$ of variables be equal to L and the length of the tuple $\overline{w}$ be equal to N.

For a tuple $\overline{u}$ we denote by $\mathrm{Set}(\overline{u})$ the set of all elements of $\overline{u}$.

In what follows, we consider L-formulae $\theta_{l_1,\dots,l_k}(x, \overline{y}, \overline{u}, \overline{v})$ in which

$$\begin{gathered}\overline{u} = u_{1,1}, \dots, u_{1,l_1}, \dots, u_{k,1}, \dots, u_{k,l_k},\\ \overline{v} = v_1, \dots, v_{k-1},\\ k, l_1, \dots, l_k \geqslant 1,\end{gathered}$$

and which are conjunctions of L-formulae asserting that

1) $u_{1,1} < \dots < u_{1,l_1} < v_1 < u_{2,1} < \dots < u_{2,l_2} < v_2 < \dots < u_{k-1,1} < \dots < u_{k-1,l_{k-1}} < v_{k-1} < u_{k,1} < \dots < u_{k,l_k}$;
2) $\phi(x, \overline{y}, \overline{w})$ holds for any $\overline{w}$ such that $\mathrm{Set}(\overline{w}) \subseteq \mathrm{Set}(\overline{u})$;
3) for any $v \in \mathrm{Set}(\overline{v})$ we have

$$\bigvee_{\substack{\mathrm{Set}(\overline{w})\subseteq\\ \mathrm{Set}(\overline{u})}} \bigvee_{i=1}^{N} \neg\phi(x, \overline{y}, \langle v \rightarrow_i \overline{w}\rangle).$$

Here $\langle v \rightarrow_i \overline{w}\rangle$ stands for the tuple obtained from $\overline{w}$ by replacing w_i by v. The formula in 3) means that for any $v \in \mathrm{Set}(\overline{v})$ there is a tuple $\overline{w}$ formed by elements of $\mathrm{Set}(\overline{u})$ such that $\neg\phi(x, \overline{y}, \langle v \rightarrow_i \overline{w}\rangle)$ holds for some $i \in \{1, \dots, N\}$.

Definition 27. A formula $\theta_{l_1,\dots,l_k}(x, \overline{y}, \overline{u}, \overline{v})$ is called a *generalized (k,l)-formula* if the numbers $l_1, \dots, l_k$ different from 1 are not less than l and at least one of the numbers $l_1, \dots, l_k$ exceeds 1. A generalized (k,l)-formula is called a *(k,l)-formula* if the numbers $l_1, \dots, l_k$ exceeding 1 are equal to l.

The number k is called the *number of series* in the generalized (k,l)-formula. For $i \in \{1, \dots, k\}$ the tuple $\overline{u}_i = u_{i,1}, \dots, u_{i,l_i}$ is called a *series.* If $l_i = 1$, then the series is called a *point series.*

Let the formula

$$\theta_{l_1,\dots,l_k}(x_0, \overline{y}_0, \overline{a}, \overline{b})$$

be true in M for elements $\overline{a} \in I$, $\overline{b} \in I$, x_0, and $\overline{y}_0$, where

$$\overline{a} = a_{1,1}, \dots, a_{1,l_1}, \dots, a_{k,1}, \dots, a_{k,l_k}.$$

Then for $i \in \{1, \dots, k\}$ the tuple

$$\overline{a}_i = a_{i,1}, \dots, a_{i,l_i}$$

is also called a *series.*

Let

$$[\overline{a}_i] = \{d \in I \mid a_{i,1} \leqslant d \leqslant a_{i,l_i}\},$$

$$[\overline{a}]_{\theta_{l_1,\dots,l_k}} = \bigcup_{i=1}^{k} [\overline{a}_i].$$

Strictly speaking, a partition of the tuple $\overline{a}$ into series, and hence $[\overline{a}]_{\theta_{l_1,\dots,l_k}}$ as well, depends on the formula $\theta_{l_1,\dots,l_k}$. However, in what follows we omit the subscript $\theta_{l_1,\dots,l_k}$ and simply write $[\overline{a}]$ if the formula meant is clear from the context.

The following lemma shows that the number of series in a satisfiable generalized (k,l)-formula cannot be too large.

Lemma 8.8. *If for elements $\overline{a} \in I$, $\overline{b} \in I$, x_0, and $\overline{y}_0$ the formula*

$$\theta_{l_1,\dots,l_k}(x_0, \overline{y}_0, \overline{a}, \overline{b})$$

is true in M and the formula

$$(\forall \overline{w} \in P)(\psi_\phi(\overline{c}, \overline{w}) \leftrightarrow \phi(x_0, \overline{y}_0, \overline{w}))$$

is true in (M, I), then in any segment $[b_i, a_{i+N,1}]$ there is at least one element belonging to the tuple $\overline{c}$. Hence, the number of series in a satisfiable generalized (k,l)-formula does not exceed $N(L+1)$.

Proof. Let b_i be an arbitrary element of the tuple $\overline{b}$. We choose a tuple $\overline{d} \in \mathrm{Set}(\overline{a})$ such that

$$(M, I) \models \neg\phi(x_0, \overline{y}_0, \langle b_i \underset{j}{\rightarrow} \overline{d}\,\rangle)$$

for some j.

Let us change the elements of the tuple $\langle b_i \rightarrow_j \overline{d}\,\rangle$ in such a way that the old mutual ordering of the elements of the tuple is preserved, the elements outside the segment $[b_i, a_{i+N,1}]$ are preserved, and the modified tuple is formed only by elements of the set $\mathrm{Set}(\overline{a})$. If we can do this, then the lemma will follow from Remark 8.5.

We replace the element b_i by $a_{i+1,1}$. If $a_{i+1,1}$ already appears in $\overline{d}$, then we replace it by the next element of the set $\mathrm{Set}(\overline{a})$. If this element also appears in $\overline{d}$, then we replace it by the next element of the set $\mathrm{Set}(\overline{a})$, and so on. Since the length of $\overline{d}$ is equal to N, we make at most N replacements.

Let Q be $(N(L+1)+1)(N-1)+2$.

A generalized (k, Q)-formula is called a *generalized k-formula*, and a (k, Q)-formula is called a *k-formula*. Since there are at most 2^k distinct k-formulae for a given k, it follows that for all values of k there are at most $2^{N(L+1)+1} - 1$ distinct satisfiable k-formulae.

Let us enumerate all these formulae by the natural numbers

$$0, 1, 2, \ldots, K$$

as follows. We first index the formulae with a single series, then those with two series, and so on. Thus, any formula with a greater number of series has a greater index. The formulae with equally many series are enumerated in such a way that the formulae with a greater number of point series are enumerated first, and formulae with equally many series and equally many point series are enumerated in an arbitrary way. We denote the formula with index i by $\gamma_i(x, \overline{y}, \overline{u}, \overline{v})$.

Lemma 8.9 (on a finite set). *For any generalized k-formula*

$$\theta_{l_1,\ldots,l_k}(x, \overline{y}, \overline{u}, \overline{v}),$$

any tuples $\overline{a} \in I$, $\overline{b} \in I$, x_0, $\overline{y}_0$ for which the formula

$$\theta_{l_1,\ldots,l_k}(x_0, \overline{y}_0, \overline{a}, \overline{b})$$

is true in M, and any finite set G formed by elements of the set I and such that the formula $\phi(x_0, \overline{y}_0, \overline{g})$ is true in M for any $\overline{g}$ for which $\mathrm{Set}(\overline{g}) \subseteq (G \cup \mathrm{Set}(\overline{a}))$ *there exist a generalized k'-formula*

$$\theta_{l'_1,\ldots,l'_{k'}}(x, \overline{y}, \overline{u}', \overline{v}')$$

and tuples $\overline{a}' \in I$ and $\overline{b}' \in I$ such that

$$\theta_{l'_1,\ldots,l'_{k'}}(x_0, \overline{y}_0, \overline{a}', \overline{b}')$$

is true in M and $(\mathrm{Set}(\overline{a}) \cup G) \subseteq \mathrm{Set}(\overline{a}')$. *Moreover, $k' \geqslant k$. If $k' = k$, then every series is either extended or preserved.*

Proof. As the first step, we include every element g of the set G in a series $\overline{a}$ such that there are no elements of the form b_i between $\overline{a}$ and g.

After this, it can turn out that the lengths of some non-point series are less than Q. We must get rid of these series. We do this in several steps.

At any step we consider an arbitrary 'short' series $\overline{a}_i$. If there is an element $h \in I$ in $[\overline{a}_i]$ such that for some $\overline{d} \in \mathrm{Set}(\overline{a})$ we have $\neg\phi(x_0, \overline{y}_0, \langle h \rightarrow_n \overline{d} \rangle)$ for some n, then we include this element h into a new $\overline{b}$ and split the series $\overline{a}_i$ into two shorter series. There can be at most $N(L+1)$ steps of this kind. After completing all these steps we see that every element in $[\overline{a}_i]$ can be adjoined to the series $\overline{a}_i$.

Lemma 8.10 (on inner elements). *For any generalized k-formula*

$$\theta_{l_1,\dots,l_k}(x,\overline{y},\overline{u},\overline{v})$$

and any tuples $\overline{a} \in I$, $\overline{b} \in I$, x_0, $\overline{y}_0$ for which the formula

$$\theta_{l_1,\dots,l_k}(x_0,\overline{y}_0,\overline{a},\overline{b})$$

is true in M there exist a k'-formula

$$\theta_{l'_1,\dots,l'_{k'}}(x,\overline{y},\overline{u}',\overline{v}')$$

and tuples $\overline{a}' \in I$ and $\overline{b}' \in I$ such that

$$\theta_{l'_1,\dots,l'_{k'}}(x_0,\overline{y}_0,\overline{a}',\overline{b}')$$

is true in M, $\mathrm{Set}(\overline{a}) \subseteq [\overline{a}']$, *and the formula $\phi(x_0,\overline{y}_0,\overline{e})$ holds in M for any tuple $\overline{e}$ such that* $\mathrm{Set}(\overline{e}) \subseteq [\overline{a}']$. *Moreover, $k' \geqslant k$. If $k' = k$, then the formula $\phi(x_0,\overline{y}_0,\overline{e})$ holds in M for any tuple $\overline{e}$ such that* $\mathrm{Set}(\overline{e}) \subseteq [\overline{a}]$.

Proof. Let us first construct a generalized k'-formula. If the formula $\phi(x_0,\overline{y}_0,\overline{e})$ holds in M for any tuple $\overline{e}$ such that $\mathrm{Set}(\overline{e}) \subseteq [\overline{a}]$, then a good tuple is given by

$$\theta_{l_1,\dots,l_k}(x,\overline{y},\overline{u},\overline{v})$$

together with $\overline{a}$ and $\overline{b}$.

Let $\mathrm{Set}(\overline{e}) \subseteq [\overline{a}]$ and let $\neg\phi(x_0,\overline{y}_0,\overline{e})$ be true in M. We successively adjoin to $\overline{a}$ the elements e in $\overline{e}$ for which the formula

$$\phi(x_0,\overline{y}_0,\langle e \rightarrow_n \overline{d}\,\rangle)$$

is true for any n for any $\overline{d}$ with $\mathrm{Set}(\overline{d}) \subseteq \mathrm{Set}(\overline{a})$. Here some series become extended. Obviously, it is impossible to adjoin all elements of $\overline{e}$. Hence, at some step, for any e among the remaining elements of $\overline{e}$ there is a $\overline{d}$ such that $\mathrm{Set}(\overline{d}) \subseteq \mathrm{Set}(\overline{a})$ and the formula $\neg\phi(x_0,\overline{y}_0,\langle e \rightarrow_n \overline{d}\,\rangle)$ holds in M for some n. We choose such an e and include it in a new $\overline{b}$. Here the number of series increases. If the formula $\phi(x_0,\overline{y}_0,\overline{e})$ holds in M for the new tuples $\overline{a}$ and $\overline{b}$ and for any tuple $\overline{e}$ such that $\mathrm{Set}(\overline{e}) \subseteq [\overline{a}]$, then the construction is completed. Otherwise, we choose a tuple $\overline{e}$ such that $\mathrm{Set}(\overline{e}) \subseteq [\overline{a}]$ and $\neg\phi(x_0,\overline{y}_0,\overline{e})$ is true in M, and we repeat the construction. After at most $N(L+1)$ steps of this kind we arrive at the case in which the formula $\phi(x_0,\overline{y}_0,\overline{e})$ holds in M for the new tuples $\overline{a}$ and $\overline{b}$ and for any tuple $\overline{e}$ such that $\mathrm{Set}(\overline{e}) \subseteq [\overline{a}]$. Moreover, some non-point series can turn out to be short; however, by extending a short series $[\overline{a}_i]$ of this kind by sufficiently many elements $e \in [\overline{a}_i]$, we can make the number of elements of this series to be not less than Q.

We now show how to reduce any series containing more than Q elements to a series with Q elements.

For any $b_i \in \mathrm{Set}(\overline{b})$ at most $N-1$ elements used of the series can appear in any tuple $\overline{d}$ such that $\mathrm{Set}(\overline{d}) \subseteq \mathrm{Set}(\overline{a})$ and the formula $\neg\phi(x_0,\overline{y}_0,\langle b_i \rightarrow_n \overline{d}\,\rangle)$ holds in M for some n. Hence, one needs at most $(N(L+1)+1)(N-1)$ elements of this kind in the series. We also need two extreme elements of the series. In all, we need at most $(N(L+1)+1)(N-1)+2$ elements. The other elements can be deleted. This proves Lemma 8.10.

Combining the last two lemmas, we obtain the following result.

Lemma 8.11. *For any generalized k-formula*

$$\theta_{l_1,\dots,l_k}(x,\overline{y},\overline{u},\overline{v}),$$

any tuples $\overline{a}\in I$, $\overline{b}\in I$, x_0, $\overline{y}_0$ for which the formula

$$\theta_{l_1,\dots,l_k}(x_0,\overline{y}_0,\overline{a},\overline{b})$$

is true in M, and any finite set G composed of elements of I such that the formula $\phi(x_0,\overline{y}_0,\overline{g})$ is true in M for any $\overline{g}$ with $\mathrm{Set}(\overline{g})\subseteq(G\cup\mathrm{Set}(\overline{a}))$ there exist a k'-formula

$$\theta_{l'_1,\dots,l'_{k'}}(x,\overline{y},\overline{u}',\overline{v}')$$

and tuples $\overline{a}'\in I$ and $\overline{b}'\in I$ such that the formula

$$\theta_{l'_1,\dots,l'_{k'}}(x_0,\overline{y}_0,\overline{a}',\overline{b}')$$

is true in M, $(\mathrm{Set}(\overline{a})\cup G)\subseteq[\overline{a}']$, and the formula $\phi(x_0,\overline{y}_0,\overline{e})$ holds in M for any tuple $\overline{e}$ such that $\mathrm{Set}(\overline{e})\subseteq[\overline{a}']$. Moreover, $k'\geqslant k$. If $k'=k$, then for any tuple $\overline{e}$ such that $\mathrm{Set}(\overline{e})\subseteq[\overline{a}]$ the formula $\phi(x_0,\overline{y}_0,\overline{e})$ holds in M.

We are now ready to construct the formulae

$$\eta_i(x,\overline{y},\overline{u}^i,\overline{v}^i)$$

by backwards induction on i. As

$$\eta_K(x,\overline{y},\overline{u}^K,\overline{v}^K)$$

we take $\gamma_K(x,\overline{y},\overline{u}^K,\overline{v}^K)$. Since $\gamma_i(x,\overline{y},\overline{u}^i,\overline{v}^i)$ is an L-formula for any i, it follows that the formula

$$(\exists\, x)\eta_K(x,\overline{y},\overline{u}^K,\overline{v}^K)$$

is an L-formula, and thus a P-bounded formula. Suppose that the formulae

$$\eta_j(x,\overline{y},\overline{u}^j,\overline{v}^j)$$

have already been constructed for $j=K,\ \dots,\ i+1$ in such a way that for $j=K,\dots,i+1$ the formula

$$(\exists\, x)\eta_j(x,\overline{y},\overline{u}^j,\overline{v}^j)$$

is equivalent to a P-bounded formula. This means that for any $j=K,\dots,i+1$ there is a quantifier-free order formula

$$\psi_j(\overline{f}_j,\overline{u}^j,\overline{v}^j),$$

such that

$$(\forall\,\overline{y})(\exists\,\overline{f}_j\in P)(\forall\,\overline{u}^j,\overline{v}^j\in P)((\exists\, x)\eta_j(x,\overline{y},\overline{u}^j,\overline{v}^j)\leftrightarrow\psi_j(\overline{f}_j,\overline{u}^j,\overline{v}^j)).$$

Suppose that the length of a tuple $\overline{e}_j$ is equal to $(n_j+1)(N(L+1)+1)$, where n_j is the length of $\overline{f}_j$. Let a quantifier-free order formula

$$\Theta_j(\overline{f}_j, \overline{e}_j)$$

assert that each of the open intervals into which the tuple $\overline{f}_j$ partitions I (for convenience, we refer to these intervals as $\overline{f}_j$-intervals) contains at least $N(L+1)+1$ distinct elements of $\overline{e}_j$.

Let

$$\eta_i(x, \overline{y}, \overline{u}^i, \overline{v}^i)$$

be the following formula:

$$\begin{aligned}(\exists\, \overline{f}_K, \ldots, \overline{f}_{i+1} \in P)\Bigg(\bigg(\bigwedge_{j=i+1}^{K} (\forall\, \overline{u}^j, \overline{v}^j \in P)\\ ((\exists\, x)\eta_j(x, \overline{y}, \overline{u}^j, \overline{v}^j) \leftrightarrow \psi_j(\overline{f}_j, \overline{u}^j, \overline{v}^j))\bigg) \wedge (\exists\, \overline{e}_K, \ldots, \overline{e}_{i+1} \in P)\\ \bigg(\bigg(\bigwedge_{j=i+1}^{K} \Theta_j(\overline{f}_j, \overline{e}_j)\bigg) \wedge \bigg(\bigwedge_{\substack{\mathrm{Set}(\overline{g})\subseteq\\ (G_i\cup\mathrm{Set}(\overline{u}^i))}} \phi(x, \overline{y}, \overline{g})\bigg) \wedge \gamma_i(x, \overline{y}, \overline{u}^i, \overline{v}^i)\bigg)\Bigg),\end{aligned}$$

where the set G_i is formed by all the variables of the tuples $\overline{f}_K, \overline{e}_K, \ldots, \overline{f}_{i+1}, \overline{e}_{i+1}$.

It is clear that the formula $(\exists\, x)\eta_i$ is equivalent to a P-bounded formula.

Let $\Phi_i(\overline{y})$ be

$$\begin{aligned}&(\exists\, \overline{f}_i \in P)((\forall\, \overline{u}^i, \overline{v}^i \in P)((\exists\, x)\eta_i(x, \overline{y}, \overline{u}^i, \overline{v}^i) \leftrightarrow \psi_i(\overline{f}_i, \overline{u}^i, \overline{v}^i))\\ &\quad \wedge (\exists\, \overline{e}_i \in P)(\Theta_i(\overline{f}_i, \overline{e}_i) \wedge (\exists\, \overline{u}^i, \overline{v}^i \in P)(\exists\, x)(F_i \subseteq [\overline{u}^i]_{\gamma_i} \wedge \eta_i(x, \overline{y}, \overline{u}^i, \overline{v}^i)))),\end{aligned}$$

where the symbol F_i stands for the set formed by all elements of the tuples $\overline{f}_i$ and $\overline{e}_i$. One can obviously represent the abbreviation $F_i \subseteq [\overline{u}^i]_{\gamma_i}$ in the form of a quantifier-free order formula.

Finally, let $\Phi(\overline{y})$ be

$$\bigvee_{j=0}^{K} \Phi_j.$$

Obviously, all the formulae Φ_i, and hence the formula Φ, are equivalent to some P-bounded formulae.

Lemma 8.12. *Suppose that the formula $\Phi_i(\overline{y}_0)$ is true in (M, I) and the formula $\Phi_j(\overline{y}_0)$ is false in (M, I) for any $j > i$ for a given tuple $\overline{y}_0$ such that the formula $\Phi(\overline{y}_0)$ is true in (M, I). Let $\overline{f}_i$, $\overline{e}_i$, x_0, $\overline{a} \in I$, and $\overline{b} \in I$ be chosen in such a way that*

$$\begin{aligned}&((\forall\, \overline{u}^i, \overline{v}^i \in P)((\exists\, x)\eta_i(x, \overline{y}_0, \overline{u}^i, \overline{v}^i) \leftrightarrow \psi_i(\overline{f}_i, \overline{u}^i, \overline{v}^i))\\ &\qquad \wedge \Theta_i(\overline{f}_i, \overline{e}_i) \wedge F_i \subseteq [\overline{a}]_{\gamma_i} \wedge \eta_i(x_0, \overline{y}_0, \overline{a}, \overline{b}))\end{aligned} \tag{5}$$

is true in (M, I).

Then for any $\overline{a}' \in I$ *and* $\overline{b}' \in I$ *arranged with respect to* $\overline{f}_i$ *just like* $\overline{a}$ *and* $\overline{b}$ *there is an* x_0' *such that*

$$(M, I) \models \eta_i(x_0', \overline{y}_0, \overline{a}', \overline{b}'),$$

and the formula $\phi(x_0', \overline{y}_0, \overline{g})$ *is true in* M *for any* $\overline{g} \in [\overline{a}']_{\gamma_i}$.

Proof. Since

$$(\forall\, \overline{u}^i, \overline{v}^i \in P)((\exists\, x)\eta_i(x, \overline{y}_0, \overline{u}^i, \overline{v}^i) \leftrightarrow \psi_i(\overline{f}_i, \overline{u}^i, \overline{v}^i))$$

is true in (M, I), there is an x_0' for which

$$(M, I) \models \eta_i(x_0', \overline{y}_0, \overline{a}', \overline{b}').$$

In particular, for any $\overline{g} \in (G_i \cup \mathrm{Set}(\overline{a}'))$ the formula $\phi(x_0', \overline{y}_0, \overline{g})$ is true in M.

Let us consider an arbitrary element $\overline{g} \in [\overline{a}']_{\gamma_i}$. Suppose that for $\overline{g}$ the formula

$$\neg\phi(x_0', \overline{y}_0, \overline{g})$$

is true in M. Then by Lemma 8.11 there exist a $j > i$ and $\overline{a}''$ and $\overline{b}''$ such that:

1) $\gamma_j(x_0', \overline{y}_0, \overline{a}'', \overline{b}'')$ is true in M;
2) $(G_i \cup \mathrm{Set}(\overline{a}')) \subseteq [\overline{a}'']_{\gamma_j}$;
3) for any $\overline{g}$ such that $\mathrm{Set}(\overline{g}) \subseteq [\overline{a}'']_{\gamma_j}$ the formula $\phi(x_0', \overline{y}_0, \overline{g})$ is true in M.

We consider an arbitrary element $g \in (G_j \cup F_j \cup \mathrm{Set}(\overline{a}''))$. Since $j > i$, it follows that $(G_j \cup F_j) \subseteq G_i$. Hence, g is an element of the set $G_i \cup \mathrm{Set}(\overline{a}'')$. In this case, $g \in [\overline{a}'']_{\gamma_j}$. Therefore, the formula $\phi(x_0', \overline{y}_0, \overline{g})$ is true in M for any $\overline{g}$ such that $\mathrm{Set}(\overline{g}) \subseteq (G_j \cup \mathrm{Set}(\overline{a}''))$. This implies that $\eta_j(x_0', \overline{y}_0, \overline{a}'', \overline{b}'')$ is true in (M, I). However, this implies that Φ_j is true in (M, I), which contradicts the choice of i. This proves Lemma 8.12.

We now proceed to prove that Φ is a P-bounded formula which is equivalent to

$$(\exists\, x)(\forall\, \overline{w} \in P)\phi(x, \overline{y}, \overline{w})$$

in (M, I).

Since γ_0 has only one series, it follows that the formula $\Phi_0(\overline{y})$ needs only the following existence condition: for any given finite set $S \subseteq I$ there is an x such that the formula $\phi(x, \overline{y}, \overline{w})$ is true in M for all tuples $\overline{w}$ whose elements are taken from S. Therefore, in (M, I) it follows from

$$(\exists\, x)(\forall\, \overline{w} \in P)\phi(x, \overline{y}, \overline{w})$$

that $\Phi_0(\overline{y})$ is true, and hence $\Phi(\overline{y})$ is true.

Lemma 8.13. *It follows from* $\Phi(\overline{y})$ *in* (M, I) *that*

$$(\exists\, x)(\forall\, \overline{w} \in P)\phi(x, \overline{y}, \overline{w}).$$

Proof. Suppose that $\Phi_i(\overline{y}_0)$ is true in (M, I) and $\Phi_j(\overline{y}_0)$ is false in (M, I) for any $j > i$ for a given tuple $\overline{y}_0$ such that $\Phi(\overline{y}_0)$ is true in (M, I). Let $\overline{f}_i$, $\overline{e}_i$, x_0, $\overline{a} \in I$ and $\overline{b} \in I$ be chosen in such a way that the formula (5) is true in (M, I).

By Remark 8.7, it suffices to find an element x_0 for a given tuple $\overline{y}_0$ and a given finite set $S \subseteq I$ such that $\phi(x_0, \overline{y}_0, \overline{w})$ is true in M for all tuples $\overline{w}$ whose elements are taken in S.

Since between neighbouring elements of $\overline{f}_i$ there are at least $N(L+1)+1$ distinct elements of $\overline{e}_i$ and the number of series does not exceed $N(L+1)$, it follows that at least two elements are covered by the same series $\overline{a}$. Thus, in any $\overline{f}_i$-interval there are elements which are covered by the same non-point series $\overline{a}$.

By Lemma 8.12, for any $\overline{a}' \in I$ and $\overline{b}' \in I$ arranged with respect to $\overline{f}_i$ just like $\overline{a}$ and $\overline{b}$ there is an x_0' such that

$$(M, I) \models \eta_i(x_0', \overline{y}_0, \overline{a}', \overline{b}')$$

and $\phi(x_0', \overline{y}_0, \overline{g})$ is true in M for any $\overline{g} \in [\overline{a}']$.

By definition, $\mathrm{Set}(\overline{f}_i) \subseteq [\overline{a}]_{\gamma_i}$. If $s \in S$ does not appear in $[\overline{a}]$, then $s \notin \mathrm{Set}(\overline{f}_i)$ and the $\overline{f}_i$-interval containing s contains an extreme element of some non-point series in $\overline{a}$. Since S is finite, there are tuples $\overline{a}' \in I$ and $\overline{b}' \in I$ arranged with respect to $\overline{f}_i$ just like $\overline{a}$ and $\overline{b}$ and satisfying the condition $S \subseteq [\overline{a}']$.

9. Active queries. Collapse of a locally generic query into an active query for reducible theories

In this section we consider a small model (M, I) of a reducible theory of signature L. By Theorem 8.1, this small model is P-bounded. We also consider databases with the scheme ρ. All queries under consideration are assumed to be Boolean.

Let a formula $\mathrm{AD}(x)$ determine an active domain of the state of a database with a scheme ρ. Clearly, one can assume that $\mathrm{AD}(x)$ is a ρ-formula.

We recall that a query is said to be *active* if it is given by a formula in which all quantifiers are restricted to the active domain. Such formulae are also said to be *active*.

In more detail, the quantifier-free formulae are active. If Φ is an active formula, then

$$(\exists\, x)(\Phi \wedge \mathrm{AD}(x)) \quad \text{and} \quad (\forall\, x)(\mathrm{AD}(x) \rightarrow \Phi)$$

are also active formulae. Every active formula can be obtained by using these two rules.

Lemma 9.1. *Every query given by a P-bounded $(<, \rho, P)$-formula is equivalent in (M, I) to some active query with respect to finite states over I. This active query can be effectively constructed from the given query.*

Proof. We consider two kinds of variables. The variables of the first kind take values in the active domain and the variables of the second kind take values in the part of I that is outside the active domain. Clearly, one can assume that the quantifiers in the formula defining the query under consideration are taken with respect to variables of these two kinds.

Every quantifier-free query is active. When quantification is applied to an active formula with respect to a variable of the first kind, the formula remains active.

Thus, one can consider the case in which an active formula is subjected to existential quantification with respect to a variable y of the second kind.

We first transform the active formula itself. All atomic formulae involving a variable of the second kind y are order formulae. Since the active domain is finite, it follows that y can be less than the least element of the active domain, or it can be greater than the largest element, or the active domain can contain elements ${}_yx < x_y$ that are extreme for y and such that there are no elements of the active domain between ${}_yx$ and x_y and ${}_yx < y < x_y$. These three cases can be treated similarly, and we consider only the last of them. Every inequality $y < x$ for a variable x of the first kind in the active formula can be replaced by $x_y \leqslant x$ and every inequality $x < y$ can be replaced by $x \leqslant {}_yx$, which gives a formula which we denote by ψ. We replace the active formula itself by the formula

$$(\exists\, x_y)(\exists_y x)({}_yx < y < x_y \wedge (\forall\, x')(x' \leqslant {}_yx \vee x_y \leqslant x') \wedge \psi).$$

Here the variables x_y, ${}_yx$, and x' are of the first kind. Going through all possible orderings of the variables of the second kind used, we choose for any such ordering the greatest variable z_1 among the variables less than y and the least variable z_2 among the variables greater than y. Thus, we must find an element y such that $z_1 < y < z_2$ and ${}_yx < y < x_y$. The order on I is dense, so to find such an element y, it is necessary and sufficient that the intervals (z_1, z_2) and $({}_yx, x_y)$ intersect. This remark enables one to delete the variable y from the formula

$$(\exists\, x_y)(\exists_y x)({}_yx < y < x_y \wedge (\forall\, x')(x' \leqslant {}_yx \vee x_y \leqslant x') \wedge \psi).$$

Lemma 9.2. *Every extended query is equivalent in (M, I) for finite states over I to a query of the form*

$$(\exists\, \overline{c} \in P)(\psi(\overline{c}) \wedge \theta(\overline{c})),$$

in which $\psi(\overline{c})$ is an (L, P)-formula and $\theta(\overline{c})$ is a P-bounded $(<, \rho, P)$-formula. If (M, I) is effectively P-reducible, then the equivalent query can be effectively constructed from the given query.

Proof. We consider variables of two kinds. The variables of the first kind take values in the set I, while the values of the variables of the second kind do not belong to I. We can assume that every quantifier is taken over some variable of a definite kind. The variables of the second kind take no values in the active domain and do not appear in atomic formulae of the form $R(\overline{x})$ for $R \in \rho$. The subformulae of the form $R(\overline{x}, \overline{y})$ for $R \in L$, where the variables $\overline{x}$ are taken in I and $\overline{y}$ does not belong to I, must be replaced by the formulae $\psi_R(\overline{x}, \overline{z}_{\overline{y}})$ by taking the conjunction of the formula obtained and all the formulae

$$(\forall\, \overline{x} \in P)(\psi_R(\overline{x}, \overline{z}_{\overline{y}}) \leftrightarrow R(\overline{x}, \overline{y}))$$

and by applying all blocks of quantifications $(\exists\, \overline{z}_{\overline{y}} \in P)$ to the conjunction. After this, every quantifier-free formula takes the form

$$(\exists\, \overline{c} \in P)(\psi(\overline{c}, \overline{x}, \overline{y}) \wedge \theta(\overline{c}, \overline{x}, \overline{y})),$$

where $\psi(\overline{c}, \overline{x}, \overline{y})$ is an (L, P)-formula and $\theta(\overline{c}, \overline{x}, \overline{y})$ is a P-bounded $(<, \rho, P)$-formula. The negation of a formula of this form becomes

$$(\forall\, \overline{c} \in P)(\neg\psi(\overline{c}, \overline{x}, \overline{y}) \vee \neg\theta(\overline{c}, \overline{x}, \overline{y})).$$

Replacing $\neg\psi(\overline{c}, \overline{x}, \overline{y})$ by $\xi(\overline{c}, \overline{x}, \overline{d})$, taking the conjunction of the resulting formula and the formula

$$(\forall\, \overline{w} \in P)(\xi(\overline{w}, \overline{d}) \leftrightarrow \neg\psi(\overline{w}, \overline{y})),$$

and applying the block of quantifications $(\exists\, \overline{d} \in P)$ to this conjunction, we obtain a formula of the desired type, and this formula is equivalent to the negation of the formula under consideration. Applying existential quantification to this formula with respect to a variable of the first kind preserves the form of the formula, and applying existential quantification with respect to a variable of the second kind reduces to the same quantification applied to $\psi(\overline{c}, \overline{x}, \overline{y})$.

Lemma 9.3. *Every extended query is equivalent in (M, I) for finite states over I to a query η given by a P-bounded $(<, \rho, P)$-formula. If I is effectively indiscernible in (M, I) and (M, I) is effectively P-reducible, then η can be effectively constructed from a given query.*

Proof. Since I is an indiscernible sequence in (M, I) by Theorem 6.7, it follows that the formula $\psi(\overline{c})$ is equivalent to some quantifier-free order formula $\gamma(\overline{c})$.

Corollary 9.4. *Every extended query in (M, I) is equivalent for finite states over I to an active query θ in which θ is a $(<, \rho)$-formula. If I is effectively indiscernible in (M, I) and (M, I) is effectively P-reducible, then θ can be effectively constructed from the given query.*

Proof. We must apply Lemma 9.1.

Theorem 9.5. *If I is effectively indiscernible in (M, I) and (M, I) is effectively P-reducible, then every locally generic extended query is equivalent in (M, I) for finite states to an active restricted query, and this active query can be effectively constructed from the given extended query.*

Lemma 9.6. *If for a universe U of signature L without an independent formula there is an algorithm which determines for any closed L-formula whether or not this formula is true in U, then the small model (M, I) for $\mathrm{Th}(U)$ is effectively P-reducible.*

Proof. As was noted at the beginning of the proof of Theorem 7.2, it suffices to effectively construct a map taking every L-formula $\phi(\overline{x}, \overline{y})$ to a number n such that any tuples

$$\overline{a}_1, \ldots, \overline{a}_n$$

of values for the tuple $\overline{x}$ of variables in the support of the structure M, and all the more so in I, fail to satisfy the following condition:

(A) for any $\eta \subseteq \{1, \ldots, n\}$ there is a tuple $\overline{b}_\eta$ of values for the tuple $\overline{y}$ of variables such that

$$\eta = \{i \in \{1, \ldots, n\} \mid M \models \phi(\overline{a}_i, \overline{b}_\eta)\}.$$

Since U and M are elementarily equivalent, it suffices to effectively find an n for which any tuples

$$\overline{a}_1, \ldots, \overline{a}_n$$

of values for the tuple $\overline{x}$ of variables in the support of the structure U do not satisfy the condition (A).

Since for a given n this condition is defined by an L-formula

$$\gamma_n(\overline{a}_1, \ldots, \overline{a}_n),$$

it follows that one must find an n such that the formula

$$(\forall\, \overline{a}_1, \ldots, \overline{a}_n) \neg \gamma_n(\overline{a}_1, \ldots, \overline{a}_n)$$

is true. Beginning with $n = 1$, we look through the numbers n until we find the desired value.

Thus, it remains to effectively construct an indiscernible sequence. By the Malcev compactness theorem (Theorem 3.5), it remains to construct an effectively almost indiscernible countable sequence in which the indiscernibility condition holds for any formula beginning at some point. One can easily construct such a sequence for the Presburger arithmetic and for the field of real numbers. For this reason, every extended query for these theories can be effectively converted into an equivalent active restricted query.

10. Theory with an independent formula

We denote by $\mathbb{N}$ the set of natural numbers. As was proved in [16], the elementary theory of the structure $(\mathbb{N}, <, +, |_p)$ is decidable, where $p > 1$ is a natural number and $x \,|_p\, y$ is defined by the formula

$$(\exists\, u)(\exists\, k)(x = p^u \wedge y = kx).$$

It is trivial (and was noted in [21]) that $(\mathbb{N}, <, +, |_p)$ admits an independent formula.

Indeed, let $x_i = p^{u_i}$ for $i = 1, \ldots, m$, where the elements $u_1, \ldots, u_m$ are pairwise distinct. Let $y = x_1 + \cdots + x_m$. In this case, the formula

$$(\exists\, u)(\exists\, v)(px \,|_p\, u \wedge v < x \wedge y = x + u + v) \tag{6}$$

holds for x if and only if $x \in \{x_1, \ldots, x_m\}$. Therefore, (6) is an independent formula.

Let the formula $f_p(x) = y$ mean that $(y \,|_p\, x \wedge \neg(py \,|_p\, x))$.

Then $(y \,|_p\, x)$ is equivalent to

$$(f_p(x) \geqslant y \wedge f_p(y) = y).$$

Therefore, the structures $(\mathbb{N}, <, +, |_p)$ and $(\mathbb{N}, <, +, f_p)$ can be interpreted in each other.

We also note that the structures $(\mathbb{N}, <, +, |_2)$ and $(\mathbb{N}, <, \sqsubseteq)$ can also be interpreted in each other, where the formula $x \sqsubseteq y$ means that

$$(\exists\, z)(\exists\, u)(y = z + x + u \wedge z < x \wedge 2x \,|_2\, u).$$

Indeed, the formula $(x \mid_2 y)$ holds if and only if

$$(x \sqsubseteq x \wedge (\forall\, v)(v < x \to v \not\sqsubseteq y)).$$

Moreover, the formula $x + y = z$ holds if and only if

$$\begin{aligned}(\exists\, u)(1 \not\sqsubseteq u \wedge (\forall\, v)(\forall\, w)((v \sqsubseteq v \wedge w \sqsubseteq w \\ \wedge\, v < w \wedge (\forall\, t)((t \sqsubseteq t \wedge v < t) \to w \leqslant t)) \\ \to ((w \sqsubseteq u \leftrightarrow ((v \sqsubseteq x \wedge v \sqsubseteq y) \vee (v \sqsubseteq x \wedge v \sqsubseteq u) \vee (v \sqsubseteq y \wedge v \sqsubseteq u))) \\ \wedge\, (v \sqsubseteq z \leftrightarrow ((v \sqsubseteq x \wedge v \sqsubseteq y \wedge v \sqsubseteq u) \\ \vee\, (v \sqsubseteq x \wedge v \not\sqsubseteq u \wedge v \not\sqsubseteq y) \vee (v \sqsubseteq y \wedge v \not\sqsubseteq u \wedge v \not\sqsubseteq x)))))).\end{aligned}$$

Finally, let us note that the relation $x < y$ can be used only for x and y for which $x \sqsubseteq x$ and $y \sqsubseteq y$. Indeed, the relation $u < v$ for arbitrary u and v can be expressed as

$$(\exists\, x)(x \sqsubseteq v \wedge x \not\sqsubseteq u \wedge (\forall\, y)(x < y \to (y \sqsubseteq u \leftrightarrow y \sqsubseteq v))).$$

Thus, we consider the structure $(\mathbb{N}, <, \sqsubseteq)$, where the relation $<$ is defined only on the set $\{x \mid x \sqsubseteq x\}$. This structure is a two-base structure in which the first support is the set of natural numbers with the succession operation and the second is the set of finite subsets of the natural numbers, and there is also an ordinary relation of inclusion of natural numbers in subsets. Therefore, the elementary theory of this two-base structure coincides with the weak monadic theory of second order of a single succession. Thus, the decidability of this elementary theory, and hence the decidability of the elementary theory of the structure $(\mathbb{N}, <, +, |_p)$, follows from the famous Rabin theorem on the decidability of the weak monadic second-order theory of *two* successions (see [22], [23]). This structure can also be regarded as the Fréchet ideal of an atomic Boolean algebra in which the set of atoms is ordered according to the type of natural numbers. In this case the role of atoms is played by the a such that $a \sqsubseteq a$. In what follows, we do refer to these elements as *atoms*.

We now claim that the collapse theorem fails for the structure $(\mathbb{N}, <, \sqsubseteq)$.

Let us consider the scheme of databases

$$\rho = \langle S, <_c, F, R, 0_c, c\rangle,$$

in which S is a unary relation symbol, F and $<_c$ are binary relation symbols, R is a unary relation symbol, and c and 0_c are the symbols of distinguished elements.

Let a ρ-formula $\mathrm{AD}_\rho(x)$ be such that for any universe U, for any $a \in U$, and for any ρ-state s in U the formula $\mathrm{AD}_\rho(a)$ is true if and only if a belongs to the active domain of the state s.

We say that all quantifiers in a formula Ψ are bounded by the formula $\mathrm{AD}_\rho(x)$ if for any subformula $(\forall\, y)\Phi$ and any subformula $(\exists\, y)\Phi$ of Ψ, where y is a variable, y differs from any bound variable of the formula $\mathrm{AD}_\rho(x)$, the formula $(\forall\, y)\Phi$ is treated as $(\forall\, y)(\mathrm{AD}_\rho(y) \to \Phi)$, and the formula $(\exists\, y)\Phi$ is treated as $(\exists\, y)(\mathrm{AD}_\rho(y) \wedge \Phi)$.

Let ϕ be the conjunction of the following ρ-sentences in which all quantifiers are bounded by the formula $\mathrm{AD}_\rho(x)$.

a) A sentence asserting that $<_c$ is a linear order on the active domain with maximal element c and minimal element 0_c.
b) A sentence asserting that S is a binary operation.
c) $(\forall x)(\forall y)(\forall u)(\forall v)(\exists w)(S(x,0,x) \wedge ((S(x,y,u) \wedge \theta(y,v)) \rightarrow (S(x,v,w) \wedge \theta(u,w))))$, where $\theta(a,b)$ is an abbreviation for

$$((a = c \wedge b = c) \vee (\forall x)(a <_c b \wedge (a <_c x \rightarrow (b <_c x \vee x = b))));$$

thus, $\theta(a,b)$ asserts that b is the element following a with respect to the ordering $<_c$ if a differs from c, and b is c if a is c; the whole formula says that S defines the addition $+_c$ on the initial segment of natural numbers that is represented by the active domain.
d) A sentence asserting that F defines a unary operation,

$$(\exists v)(F(0,v) \wedge \theta(0,v))$$

and

$$(\forall x)(\forall y)(\forall z)((F(x,y) \wedge \theta(x,z)) \rightarrow (\exists u)(\exists w_1)(\exists w_2)$$
$$(F(z,u) \wedge S(y,y,w_1) \wedge S(w_1,y,w_2) \wedge S(w_2,y,u))).$$

This formula asserts that the operation defined by F is 4^x.
e) A formula asserting that the formula $F(x,y)$ holds for any two elements x and y in R one of which immediately follows the other:

$$(\forall x)(\forall y)((R(x) \wedge R(y) \wedge x <_c y \wedge (\forall z)((x <_c z \wedge R(z))$$
$$\rightarrow (z = y \vee y <_c z))) \rightarrow F(x,y)).$$

f) $R(0_c)$ and $R(c)$.
g) The sentence asserting for any x in R with $x <_c c$, any y with $y <_c x$, and any z with $z <_c 4^y$ that the inequality

$$4^y +_c 4^y +_c 4^y +_c z <_c c$$

holds; here 4^a stands for an element of the system for which the formula $F(a, 4^a)$ holds and $(a +_c b)$ stands for an element of the system for which the formula $S(a, b, (a +_c b))$ holds.

In what follows, we consider only ρ-states s for $(\mathbb{N}, <, \sqsubseteq)$ that satisfy ϕ. Moreover, we assume that $<_c$ is the restriction of $<$ to the active domain of the state s.

We are now going to propose an extended sentence asserting for any ρ-state s that $s(R)$ is even.

Every element in $(\mathbb{N}, <, \sqsubseteq)$ can be regarded as a finite set of atoms. For any element a in the active domain of the state s distinct from $s(c)$ there is an element b of the active domain which follows the element a, and one can form the set $((a \backslash b) \cup (b \backslash a))$. Let us take a maximal element in $((a \backslash b) \cup (b \backslash a))$. This maximal atom $f_s(a)$ is uniquely determined. As $\gamma(x,y)$ we take the extended formula asserting for any x in the active domain which differs from the maximal element of the active domain that y is the maximal element in the symmetric difference of x and the element of the active domain following x. This defines a function f_s from the active domain of the state s into the natural numbers.

Lemma 10.1. *Let 1_c be an element of* $\mathrm{AD}(s)$ *following the element* 0_c *with respect to the order. For any $a \in s(R)$ distinct from $s(c)$, $s(0_c)$, and $s(1_c)$ there is a $b \in \mathrm{AD}(s)$ such that $a < b \leqslant 4^a$ and $f_s(b)$ differs from any $f_s(d)$ for any $d \in \mathrm{AD}(s)$ with $d \leqslant a$.*

Proof. Let $a \in s(R)$ and $s(1_c) < a < s(c)$. Let

$$X = \{f_s(d) \mid d \in \mathrm{AD}(s),\ d \leqslant a\}.$$

Suppose that $f_s(b) \in X$ for any $b \in \mathrm{AD}(s)$ with $a < b \leqslant 4^a$.

For any $b \in \mathrm{AD}(s)$ we consider a subset $h(b)$ of X such that $e \in h(b)$ for $e \in X$ if and only if $e \sqsubseteq b$.

If $a < b_1 < b_2 \leqslant 2^a$, $b_1 \in \mathrm{AD}(s)$, and $b_2 \in \mathrm{AD}(s)$, then $h(b_1) < h(b_2)$ in $(\mathbb{N}, <, \sqsubseteq)$.

Indeed, if b_2 is the element of the active domain of s immediately following the element b_1 with respect to $<$, then the maximal number of $((b_1 \setminus b_2) \cup (b_2 \setminus b_1))$ belongs to b_2. According to our construction, this maximal number belongs to X. Hence, in this case we have $h(b_1) < h(b_2)$ in $(\mathbb{N}, <, \sqsubseteq)$. Since the relation $<$ is transitive, it follows that $h(b_1) < h(b_2)$ if $a < b_1 < b_2 \leqslant 2^a$, $b_1 \in \mathrm{AD}(s)$, and $b_2 \in \mathrm{AD}(s)$.

Suppose that there are exactly n elements in $\mathrm{AD}(s)$ which are less than a. In this case the number of all elements of the set X does not exceed $n+1$. Thus, the number of all subsets of X does not exceed 2^{n+1}. However, the number of all elements of the set $\mathrm{AD}(s)$ that are between a and 4^a and differ from a is equal to $4^n - n$, which is greater than 2^{n+1} for $n \geqslant 2$, a contradiction.

One can easily see that, using the formula γ, one can construct a formula $\beta(x, y)$ which is true for (a, b) if b is a minimal element of the active domain and satisfies the condition of the lemma.

Theorem 10.2. *The query*

"the state s satisfies ϕ and the set $s(R)$ has an even number of elements"

is locally generic and can be given by an extended sentence.

Proof. For $a \in s(R)$ and $s(1_c)a < s(c)$ the extended formula $\beta(a, b)$ asserts that $b \in \mathrm{AD}(s)$ is minimal among all elements d such that $d \in \mathrm{AD}(s)$, $a < d \leqslant 4^a$, and the atom $f_s(d)$ differs from any atom $f_s(e)$ for any $e \in \mathrm{AD}(s)$, $e \leqslant a$. Using β, one can construct the formula

$$\alpha(x, z) = (\exists\, y)(\beta(x, y) \wedge \gamma(y, z)).$$

The formula $\alpha(x, z)$ chooses a unique atom z for any $x \notin \{0_c, 1_c, c\}$. It remains to write out a formula asserting that the set A of all atoms z chosen in this way is odd.

We propose the following formula. It asserts that there is a set Y of atoms that contains the first element of the set A, contains the next element of A if and only if Y does not contain the element of A under consideration, and contains the last element of A.

It remains to prove the following theorem.

Theorem 10.3. *There is no restricted sentence giving the query introduced in Theorem 10.2.*

The following statements follow this theorem and from Theorem 10.2.

Corollary 10.4. *The collapse theorem fails for the theory of the system* $(\mathbb{N}, <, \sqsubseteq)$.

Corollary 10.5. *There is an enrichment of the Presburger arithmetic that has a decidable elementary theory and for which the collapse theorem fails.*

To prove Theorem 10.3, one must describe the $(<, \rho)$-theories of finite ρ-states of the structure $(\mathbb{N}, <)$ that satisfy the formula ϕ. We recall that ϕ is the conjunction of the formulae a)–g). It turns out that for any $(<, \rho)$-sentence ψ one can construct a quantifier-free order formula asserting that c (regarded as a natural number) satisfies some inequalities with given natural numbers and is equivalent to ψ for finite ρ-states of the structure $(\mathbb{N}, <)$ which satisfy the formula ϕ. The details of constructing a quantifier-free order formula of this kind are routine but rather cumbersome, and we do not present them here. It is clear that such a quantifier-free order formula cannot assert that the set $s(R)$ is even.

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