Isomorphisms and strong finite projective classes of commutative semigroups

Michael A. Taitslin

ABSTRACT. In “Sverdlovsk notebook” (Sverdlovsk, 1969), I proposed a question: Are any too first-order equivalent finitely generated commutative semigroups isomorphic? In 1970, B.I.Zilber answered the question negatively. A question arises: In what language, any equivalent over the language finitely generated commutative semigroups are isomorphic? In the note, we propose such a language. Moreover, we prove that there is an algorithm which for a given finite set of generators, a given finite set of defining relations of a commutative semigroup for the generators, and a closed formula of the language decides whether the formula holds in the semigroup. [1]

1 Introduction

In [4] and “Sverdlovsk notebook” (Ural State University, Sverdlovsk, 1969) [2], I proposed a question: Are any first-order equivalent finitely generated commutative semigroups isomorphic? In [7], B.I.Zilber answered the question to construct two elementary equivalent commutative semigroups with 4 generators and 7 defining relations each. Zilber proved that the semigroups are not isomorphic.

The Zilber's example shows that the signature $\langle + \rangle$ is not relevant to describe a finite generated commutative semigroup up to isomorphism in first-order logic.

In the paper we consider an expanded signature to add unary relation G_a and constant symbol a for each generator a of the investigated finitely generated commutative semigroup and for a finite set of linear combinations of the generators. Having finite sets of generators and defining relations for the generators, we construct effectively a closed first-order formula of the expanded signature such that the formula holds in a finite generated commutative semigroup iff the new semigroup is isomorphic to the investigated semigroup. The truth of the formula in a commutative semigroup means that we can define the relations G_a and the constants a by such a way that the formula is true.

The result was formulated in [4] but the presented proof was not presented in detail. We propose a detailed proof and a new corollary.

[1]To Grigory Mints' 70-th Anniversary. The work is partially supported by the Russian Foundation of Basic Research (project code: 08-01-00241)

[2]“Sverdlovsk notebook” is a collection of problems in semigroup theory

2 Definitions

We use terminology from [5], §3. Let us recall some results and definitions. A reason is to make the paper self-sufficient.

The set $\omega = \{0, 1, 2, \dots\}$ is called the set of natural numbers. By $L(\mathfrak{A})$ we denote the free semigroup with the finite set $\mathfrak{A}$ of free generators in the class of commutative semigroups with zero. For $\mathfrak{A} = \{a_1, \dots, a_k\}$, we consider $L(\mathfrak{A})$ as the set of all linear forms on the letters $a_1, \dots, a_k$ with natural numbers as coefficients.

For $a, b \in L(\mathfrak{A})$, the pair (a, b) is called a defining relation for $\mathfrak{A}$. A semigroup $L(\mathfrak{A}, \mathfrak{B})$, given in the class of commutative semigroups with zero, by finite sets of generators $\mathfrak{A}$ and defining relations $\mathfrak{B}$, is considered as a factor semigroup of the semigroup $L(\mathfrak{A})$. For $x \in L(\mathfrak{A})$, by $\overline{x}_{\mathfrak{B}}$ we denote the image of x under the canonical mapping $L(\mathfrak{A}) \longrightarrow L(\mathfrak{A}, \mathfrak{B})$. If it is understandable what $\mathfrak{B}$ is meant, we write $\overline{x}$ instead of $\overline{x}_{\mathfrak{B}}$.

By $M(\mathfrak{A}, \mathfrak{B})$ we denote the semigroup given in the class of commutative semigroups with cancellation and with zero, by finite sets of generators $\mathfrak{A}$ and defining relations $\mathfrak{B}$. We also treat $M(\mathfrak{A}, \mathfrak{B})$ as a factor semigroup of the semigroup $L(\mathfrak{A})$. For $x \in L(\mathfrak{A})$, by $[x]_{\mathfrak{B}}$ we denote the image of x under the canonical mapping $L(\mathfrak{A}) \longrightarrow M(\mathfrak{A}, \mathfrak{B})$. If it is understandable what $\mathfrak{B}$ is meant, we write $[x]$ instead of $[x]_{\mathfrak{B}}$.

For $a \in \mathfrak{A}$ and $f \in L(\mathfrak{A})$, we denote by $(f)_a$ the coefficient of the letter a in the form f. For $f, g \in L(\mathfrak{A})$, we write $f \leqslant g$ iff $(f)_a \leqslant (g)_a$ for all $a \in \mathfrak{A}$. If $f \leqslant g$, we say that f is *less* than g if f and g are different. We say that a form $f \in S$ is *minimal* in a set $S \subseteq L(\mathfrak{A})$ of forms iff g is not less than f for any $g \in S$. We say that a form $f \in S$ is *maximal* in a set $S \subseteq L(\mathfrak{A})$ of forms iff f is not less than g for any $g \in S$.

We write $\mathfrak{A}_1 \subset \mathfrak{A}$ iff $\mathfrak{A}_1$ is a subset of $\mathfrak{A}$ and $\mathfrak{A}_1 \neq \mathfrak{A}$.

$\emptyset$ denotes the empty set. $L(\emptyset)$ denotes the zero semigroup.

If $\mathfrak{A}_1 \subset \mathfrak{A}$, we consider the semigroup $L(\mathfrak{A}_1)$ as a sub-semigroup of the semigroup $L(\mathfrak{A})$. We also suppose that $\mathfrak{A}$ is a subset of $L(\mathfrak{A})$. θ denotes zero of the semigroup $L(\mathfrak{A})$.

For $\mathfrak{A}_1 \subset \mathfrak{A}$ and $f \in L(\mathfrak{A})$, we denote by $\mathfrak{A}_1(f)$ the form in $L(\mathfrak{A} \setminus \mathfrak{A}_1)$ such that $(\mathfrak{A}_1(f))_a = (f)_a$ for any $a \in \mathfrak{A} \setminus \mathfrak{A}_1$.

Let $q(\mathfrak{B})$be the number of relations in $\mathfrak{B}$. Let $\lambda(\mathfrak{B})$ be the largest coefficient in these relations. We always assume that $\lambda(\mathfrak{B}) > 0$. By $h(\mathfrak{A}, \mathfrak{B})$ we denote the form

$$\sum_{a \in \mathfrak{A}} \lambda(\mathfrak{B})(q(\mathfrak{B}) + 1)a$$

in $L(\mathfrak{A})$.

For any $\mathfrak{A}_1 \subset \mathfrak{A}$, we denote by $\mathfrak{N}(\mathfrak{A}_1, \mathfrak{B})$ the set of all $x \in L(\mathfrak{A} \setminus \mathfrak{A}_1)$ such that the inequality

$$\overline{x + y} \neq \overline{h(\mathfrak{A}, \mathfrak{B})} + \overline{z}$$

holds in $L(\mathfrak{A}, \mathfrak{B})$ for all $y \in L(\mathfrak{A}_1)$, $z \in L(\mathfrak{A})$. By $\mathfrak{M}(\mathfrak{A}_1, \mathfrak{B})$ we denote the set of maximal elements of $\mathfrak{N}(\mathfrak{A}_1, \mathfrak{B})$. Finally, by $\mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{B})$ we denote the set

$$\{x \in L(\mathfrak{A} \setminus \mathfrak{A}_1) \mid (\exists y)(y \in \mathfrak{M}(\mathfrak{A}_1, \mathfrak{B}) \& x \leqslant y)\}.$$

It follows from the theorem of Dickson ([1], p.129) that the sets $\mathfrak{M}(\mathfrak{A}_1, \mathfrak{B})$ and $\mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{B})$ are finite.

We divide the set $\mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{B})$ into reduction classes, assigning $x, y \in \mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{B})$ to the same class iff there exists $z, u \in L(\mathfrak{A}_1)$ such that $\overline{x+z} = \overline{y+u}$ in $L(\mathfrak{A}, \mathfrak{B})$. By $i(\mathfrak{A}_1, \mathfrak{B})$ we denote the number of reduction classes, and by $\mathfrak{M}_{2,1}(\mathfrak{A}_1, \mathfrak{B})$,..., $\mathfrak{M}_{2,i(\mathfrak{A}_1,\mathfrak{B})}(\mathfrak{A}_1, \mathfrak{B})$ all the different classes.

For $j \in \{1, \ldots, i(\mathfrak{A}_1, \mathfrak{B})\}$, consider $x, y \in \mathfrak{M}_{2,j}(\mathfrak{A}_1, \mathfrak{B})$ and $\mathfrak{A}_3 \subset \mathfrak{A}_1$. We denote by $\mathfrak{N}(\mathfrak{A}_1, \mathfrak{A}_3, x, y, \mathfrak{B})$ the set of all those $z \in L(\mathfrak{A}_1 \setminus \mathfrak{A}_3)$ such that the inequality $\overline{x+z+u} \neq \overline{y+v}$ holds in $L(\mathfrak{A}, \mathfrak{B})$ for all $u \in L(\mathfrak{A}_3)$ and $v \in L(\mathfrak{A}_1)$. We denote by $\mathfrak{M}(\mathfrak{A}_1, \mathfrak{A}_3, x, y, \mathfrak{B})$ the set of maximal elements of $\mathfrak{N}(\mathfrak{A}_1, \mathfrak{A}_3, x, y, \mathfrak{B})$. By $\mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{A}_3, x, y, \mathfrak{B})$ we denote the set

$$\{u \in L(\mathfrak{A}_1 \setminus \mathfrak{A}_3) \mid (\exists v)(v \in \mathfrak{M}(\mathfrak{A}_1, \mathfrak{A}_3, x, y, \mathfrak{B}) \& u \leqslant v)\}.$$

The sets $\mathfrak{M}(\mathfrak{A}_1, \mathfrak{A}_3, x, y, \mathfrak{B})$ and $\mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{A}_3, x, y, \mathfrak{B})$ are finite (the theorem of Dickson, [1], p.129).

It was proved in [2] that

(1) the sets $\mathfrak{M}(\mathfrak{A}_1, \mathfrak{B})$, $\mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{B})$, $\mathfrak{M}(\mathfrak{A}_1, \mathfrak{A}_3, x, y, \mathfrak{B})$ are effectively constructible from $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{A}_1$, $\mathfrak{A}_3$, x, y;

(2) the set $\mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{B})$ is effectively divisible into reduction classes with respect to $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{A}_1$;

(3) there exists an effective procedure which, for $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{A}_1$, x such that $\mathfrak{A}_1 \subset \mathfrak{A}$ and $x \in L(\mathfrak{A} \setminus \mathfrak{A}_1)$, constructs a finite set $\mathfrak{B}(\mathfrak{A}_1, x)$ of defining relations for $\mathfrak{A}_1$ such that, for any $u, v \in L(\mathfrak{A}_1)$, equality $\overline{u} = \overline{v}$ holds in $L(\mathfrak{A}_1, \mathfrak{B}(\mathfrak{A}_1, x))$ iff $\overline{x+u} = \overline{x+v}$ holds in $L(\mathfrak{A}, \mathfrak{B})$;

(4) for any $x, y \in L(\mathfrak{A})$, $\overline{h(\mathfrak{A}, \mathfrak{B}) + x} = \overline{h(\mathfrak{A}, \mathfrak{B}) + y}$ holds in $L(\mathfrak{A}, \mathfrak{B})$ iff $[x] = [y]$ in $M(\mathfrak{A}, \mathfrak{B})$ (Lemma of Ceitin);

(5) for any $x \in L(\mathfrak{A})$, either there exists $z \in L(\mathfrak{A})$ such that $\overline{x} = \overline{h(\mathfrak{A}, \mathfrak{B}) + z}$ in $L(\mathfrak{A}, \mathfrak{B})$, or there exists $\mathfrak{A}_1 \subset \mathfrak{A}$ such that $\mathfrak{A}_1(x) \in \mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{B})$;

(6) for $\mathfrak{A}_1 \subset \mathfrak{A}$, $j \in \{1, \ldots, i(\mathfrak{A}_1, \mathfrak{B})\}$, $x, y \in \mathfrak{M}_{2,j}(\mathfrak{A}_1, \mathfrak{B})$, and $u \in L(\mathfrak{A}_1)$, either there exists $v \in L(\mathfrak{A}_1)$ such that $\overline{x+u} = \overline{y+v}$ in $L(\mathfrak{A}, \mathfrak{B})$, or there exist $\mathfrak{A}_3 \subset \mathfrak{A}_1$ and $v \in \mathfrak{M}(\mathfrak{A}_1, \mathfrak{A}_3, x, y, \mathfrak{B})$ such that $\mathfrak{A}_3(u) \leqslant v$.

3 A formula which claims that G_a distinguishes the sub-semigroup generated by a

LEMMA 1. *In any finite generated Abelian group, for any element a of the group, any sequence*

$$a_0, a_1, \ldots, a_n, \ldots$$

such that in the group,

$$a_i = 2a_{i+1} \vee a_i = 2a_{i+1} + a,$$

contains only finite number of different elements.

Proof. Any finite generated Abelian group is a finite direct sum of infinite cyclic groups and a finite group. Each a_i can be present as a sum of a linear combination of the generators of the infinite cyclic groups and an element of the finite group. It is enough to prove that the sequence of the linear combinations contains only finite number of different elements. So it is enough to prove for each coordinate, that there exists a natural number such that the sequence of the coordinates of the linear combinations contains only the number of different elements. But if the coordinate of a_0 is n, and the coordinate of a is k, then the module of the coordinate of a_i does not exceed $\max(|n|, |k|)$. ■

LEMMA 2. *In any finite generated commutative semigroup with cancellation, for any element a of the semigroup, any sequence*

$$a_0, a_1, \dots, a_n, \dots$$

such that in the semigroup,

$$a_i = 2a_{i+1} \vee a_i = 2a_{i+1} + a,$$

contains only finite number of different elements.

Proof. Any finite generated commutative semigroup with cancellation is isomorphically embedded into a finite generated Abelian group. ■

Having an element $\overline{a}$ of a finite presented commutative semigroup $L(\mathfrak{A}, \mathfrak{B})$ where $a \in L(\mathfrak{A})$, we denote by $\mathfrak{A}(a)_{\mathfrak{B}}$ the set of all $b \in \mathfrak{A}$ such that the equality $n\overline{a} = \overline{b} + \overline{z}$ holds in $L(\mathfrak{A}, \mathfrak{B})$ for some $z \in L(\mathfrak{A})$ and natural number n. The set $\mathfrak{A}(a)_{\mathfrak{B}}$ is closed in $L(\mathfrak{A}, \mathfrak{B})$. It means that for any $x \in L(\mathfrak{A}(a)_{\mathfrak{B}})$, and any $y, z \in L(\mathfrak{A})$, if $\overline{x} = \overline{y} + \overline{z}$ in $L(\mathfrak{A}, \mathfrak{B})$, then $y \in L(\mathfrak{A}(a)_{\mathfrak{B}})$. Thus, in this case, $\mathfrak{B}(\mathfrak{A}(a)_{\mathfrak{B}}, \theta)$ is the set of all relations from $\mathfrak{B}$ which are defining relations for $L(\mathfrak{A}(a)_{\mathfrak{B}})$.

We denote by γ_a the number $h(\mathfrak{A}(a)_{\mathfrak{B}}, \mathfrak{B}(\mathfrak{A}(a)_{\mathfrak{B}}, \theta))$.

Now we construct a formula $\kappa(a)$ of signature $\langle +, a, G_a \rangle$. We write $3a$ instead of $(a + a + a)$. The similar meaning has ia for any natural number i.

The formula $\kappa(a)$ is the conjunction of the following formulas:

(7) $(G_a(a) \& (\forall x)(\forall y)((G_a(x) \& G_a(y)) \to G_a(x + y)) \&$

$$(\forall x)((\forall y) x + y = y \to G_a(x)))$$

(it means that G_a is a sub-semigroup containing a and θ);

(8) $(\forall x)(G_a(x) \to \bigvee_{i=0}^{m-1} x = ia)$

if the sub-semigroup with zero generated by $\overline{a}$ in $L(\mathfrak{A}, \mathfrak{B})$ is finite and contains m elements;

$((\forall x)((G_a(x) \& (\exists y)(G_a(y) \& x + y = \gamma_a a)) \to \bigvee_{i=0}^{\gamma_a} x = ia) \&$

$$(\forall x)(\forall y_1)(\forall y_2)(\forall y_3)(\forall t_1)(\forall t_2)(\forall t_3)(\forall z_1)(\forall z_2)(\forall z_3)$$

$$((G_a(x) \& G_a(y_1) \& G_a(y_2) \& G_a(y_3) \& (\exists y)(G_a(y) \& x = \gamma_a a + y) \&$$

$$y_1 = t_1 + z_1 \& y_2 = t_2 + z_2 \& y_3 = t_3 + z_3 \&$$
$$x + t_1 + t_2 = x + t_1 + t_3) \to x + t_2 = x + t_3))$$

if the sub-semigroup with zero generated by $\overline{a}$ in $L(\mathfrak{A}, \mathfrak{B})$ is infinite

(it means that for infinite G_a and any natural number $\alpha \geqslant \gamma_a$, in $L(\mathfrak{A}, \mathfrak{B})$, the cancellation rule

$$((\overline{\alpha a + x + y} = \overline{\alpha a + x + z}) \to (\overline{\alpha a + y} = \overline{\alpha a + z}))$$

holds for any $x, y, z \in L(\mathfrak{A}(a)_{\mathfrak{B}})$);

(9) $(\forall x)(G_a(x) \to (\exists u)(G_a(u) \& (x = 2u + a \vee x = 2u)))$;

(10) $(\forall x)(G_a(x) \to (\exists u)(G_a(u) \& (x = u + \gamma_a a \vee \gamma_a a = x + u)))$.

The truth of $\kappa(a)$ in a finite defined commutative semigroup $L(\mathfrak{A}', \mathfrak{B}')$ means that the semigroup can be expanded to the structure of signature $\langle +, a, G_a \rangle$ such that $\kappa(a)$ holds in the expanded structure. It is obvious that $G_a(ia)$ is true for any natural number i in any expansion of $L(\mathfrak{A}', \mathfrak{B}')$ such that $\kappa(a)$ holds in the expanded structure (see (7)).

LEMMA 3. *For any $a \in L(\mathfrak{A})$, $\kappa(a)$ holds in the expansion of $L(\mathfrak{A}, \mathfrak{B})$, in which truth of $G_a(x)$ means that x is equal to $i\overline{a}$ for some natural number i and the interpretation of a is $\overline{a}$.*

Proof. (7), (9), and (10) are true in the expansion by the definition.

If the sub-semigroup with zero generated by $\overline{a}$ in $L(\mathfrak{A}, \mathfrak{B})$ is finite, the truth of (8) in the expansion is obvious.

If the sub-semigroup with zero generated by $\overline{a}$ in $L(\mathfrak{A}, \mathfrak{B})$ is infinite, the truth of (8) in the expansion is followed from the definitions of $\mathfrak{A}(a)_{\mathfrak{B}}$ and γ_a and Ceitin Lemma (see (4)). ■

LEMMA 4. *If $\kappa(a)$ holds in an expansion of $L(\mathfrak{A}', \mathfrak{B}')$, then in the expansion, truth of $G_a(x)$ implies that x is equal to ia for a natural number i.*

Proof. By contradiction. Let $G_a(x)$ holds in the expansion and x is different from ia for any natural number i. By (9), take x as u_0 and an element u_1 such that $(G_a(u_1) \& (x = 2u_1 + a \vee x = 2u_1))$. If u_j has constructed, take an element u_{j+1} such that $(G_a(u_{j+1}) \& (u_j = 2u_{j+1} + a \vee u_j = 2u_{j+1}))$. It is obvious that for any natural number j, u_j is different from ia for any natural number i. It follows from (8) and (10) that $(\exists u)(G_a(u) \& (u_j = u + \gamma_a a))$ holds in the expansion. Now it follows from (8) that $(u_j = 2u_{j+1} + a \vee u_j = 2u_{j+1})$ holds in $M(\mathfrak{A}', \mathfrak{B}')$ for some $u_0, \ldots, u_j, \ldots$ But it contradicts to Lemma 2. ■

4 A formula for equality in a finite defined commutative semigroup with cancellation

First of all, we write a first-order formula

$$\Phi_{11}(\mathfrak{A}, \mathfrak{B}, y_1, \ldots, y_k, z_1, \ldots, z_k)$$

of signature $\langle +, a, G_a \mid a \in \mathfrak{A}\rangle$ such that for any $d_1, \ldots, d_k$, $b_1, \ldots, b_k$ satisfying $G_{a_1}(d_1)$, $G_{a_1}(b_1)$, $\ldots$, $G_{a_k}(d_k)$, $G_{a_k}(b_k)$, $\Phi_{11}(\mathfrak{A}, \mathfrak{B}, d_1, \ldots, d_k, b_1, \ldots, b_k)$ implies that sums $d_1 + \cdots + d_k$ and $b_1 + \cdots + b_k$ are equal in $M(\mathfrak{A}, \mathfrak{B})$. If any generator from $\mathfrak{A}$ generates a finite sub-semigroup in $M(\mathfrak{A}, \mathfrak{B})$, $M(\mathfrak{A}, \mathfrak{B})$ is finite and $\Phi_{11}(\mathfrak{A}, \mathfrak{B}, y_1, \ldots, y_k, z_1, \ldots, z_k)$ can be taken as a first-order formula of signature $\langle +, a \mid a \in \mathfrak{A}\rangle$. The same holds if $k = 1$. Suppose $k > 1$ and $[a_1]$ generates an infinite sub-semigroup in $M(\mathfrak{A}, \mathfrak{B})$. To simplify notation, suppose that $c_i \in \mathfrak{A}$ and $(c_i, a_1 + a_i) \in \mathfrak{B}$ for $i = 2, \ldots, k$.

$\Phi_{11}(\mathfrak{A}, \mathfrak{B}, y_1, \ldots, y_k, z_1, \ldots, z_k)$ is

$$(\exists t_{1,1}) \ldots (\exists t_{1,k}) \ldots (\exists t_{q,1}) \ldots (\exists t_{q,k})$$

$$((\bigwedge_{j=1}^{q} \bigwedge_{i=1}^{k} F_{a_1,a_i}(t_{j,1}, t_{j,i}))\& \bigwedge_{i=1}^{k} [y_i + \sum_{j=1}^{q} t_{j,i}(\mathfrak{A}_{j,i} - \mathfrak{B}_{j,i}) = z_i])$$

where $\mathfrak{B}$ is $\{(\mathfrak{A}_j, \mathfrak{B}_j) \mid j = 1, \ldots, q\}$; for any $j \in \{1, \ldots, q\}$, $\mathfrak{A}_j = \sum_{i=1}^{k} \mathfrak{A}_{j,i} a_i$, and $\mathfrak{B}_j = \sum_{i=1}^{k} \mathfrak{B}_{j,i} a_i$. As usual, for natural number i, $x - iy = z$ means that $x = z + iy$. We need only explain what is F_{a_1,a_i}.

$F_{a_1,a_i}(x, y)$ tells us that there exists a natural number j such that $\overline{x}$ is equal to $j\overline{a}_1$ and $\overline{y}$ is equal to $j\overline{a}_i$ in $L(\mathfrak{A}, \mathfrak{B})$.

If $[a_i]$ generates a finite sub-semigroup in $M(\mathfrak{A}, \mathfrak{B})$, the construction of $F_{a_1,a_i}(x, y)$ is obvious.

If $[a_1]$ and $[a_i]$ are linear independent in $M(\mathfrak{A}, \mathfrak{B})$, we can use $G_{c_i}(x + y)$ instead of $F_{a_1,a_i}(x, y)$.

In other cases, there exists a natural number m such that in $M(\mathfrak{A}, \mathfrak{B})$, any sum $x = x_1 + \cdots + x_k$ where $G_{a_j}(x_j)$ for $j = 1, \ldots, k$ can be effectively presented as $x + z_1 + \cdots + z_k = y_1 + \cdots + y_k$ where $G_{a_j}(y_j)$ and $G_{a_j}(z_j)$ for $j = 1, \ldots, k$, where z_i is zero, and where $\bigvee_{j=0}^{m} y_i = j[a_i]$ (see [3]). Moreover, two such presentations $x_1 + z_{1,1} + \cdots + z_{1,k} = y_{1,1} + \cdots + y_{1,k}$ and $x_2 + z_{2,1} + \cdots + z_{2,k} = y_{2,1} + \cdots + y_{2,k}$ represent equal in $M(\mathfrak{A}, \mathfrak{B})$ elements x_1 and x_2 iff $y_{1,i}$ coincides with $y_{2,i}$ and

$$y_{1,1} + \cdots + y_{1,k} + z_{2,1} + \cdots + z_{2,k} = y_{2,1} + \cdots + y_{2,k} + z_{1,1} + \cdots + z_{1,k}.$$

It means that if $[a_i]$ generates an infinite sub-semigroup in $M(\mathfrak{A}, \mathfrak{B})$ and $[a_1]$ and $[a_i]$ are linear dependent in $M(\mathfrak{A}, \mathfrak{B})$, we do not need to use $F_{a_1,a_i}(x, y)$.

5 Main formula

Consider a finite defined commutative semigroup $L(\mathfrak{A}, \mathfrak{B})$. Now we are going to construct a closed first-order formula $\Phi(\mathfrak{A}, \mathfrak{B})$ of signature $\langle +, a, G_a \mid a \in \mathfrak{A}\rangle$ such that truth of $\Phi(\mathfrak{A}, \mathfrak{B})$ in a finite generated commutative semigroup $L(\mathfrak{A}', \mathfrak{B}')$ means that $L(\mathfrak{A}, \mathfrak{B})$ and $L(\mathfrak{A}', \mathfrak{B}')$ are isomorphic. Suppose $\mathfrak{A} = \{a_1, \ldots, a_k\}$. If any generator from $\mathfrak{A}$ generates a finite sub-semigroup in $L(\mathfrak{A}, \mathfrak{B})$, $L(\mathfrak{A}, \mathfrak{B})$ is finite and $\Phi(\mathfrak{A}, \mathfrak{B})$ can be taken as a closed first-order formula of signature $\langle + \rangle$. The same holds if $k = 1$. Suppose $k > 1$ and $\overline{a}_1$ generates an infinite sub-semigroup in $L(\mathfrak{A}, \mathfrak{B})$.

$\Phi(\mathfrak{A}, \mathfrak{B})$ is the conjunction of the formulas $\bigwedge_{a\in\mathfrak{A}} \kappa(a)$,

$$(\forall x)(\exists y_1)\dots(\exists y_k)((\bigwedge_{i=1}^{k} G_{a_i}(y_i))\&x = y_1 + \cdots + y_k),$$

and a closed first-order formula $\Phi_1(\mathfrak{A}, \mathfrak{B})$ of signature $\langle +, a, G_a \mid a \in \mathfrak{A}\rangle$.

Let us construct $\Phi_1(\mathfrak{A}, \mathfrak{B})$.

$\Phi_1(\mathfrak{A}, \mathfrak{B})$ is

$$(\forall y_1)\dots(\forall y_k)(\forall z_1)\dots(\forall z_k)((\bigwedge_{i=1}^{k} G_{a_i}(y_i)\& \bigwedge_{i=1}^{k} G_{a_i}(z_i)\&$$

$$z_1 + \cdots + z_k = y_1 + \cdots + y_k) \to \Phi_{10}(\mathfrak{A}, \mathfrak{B}, y_1, \dots, y_k, z_1, \dots, z_k)).$$

$\Phi_{10}(\mathfrak{A}, \mathfrak{B}, y_1, \dots, y_k, z_1, \dots, z_k)$ is constructed by induction on k and is

$$(((\bigwedge_{\mathfrak{A}_1\subset\mathfrak{A}} (\sum_{a_i\in(\mathfrak{A}\setminus\mathfrak{A}_1)} y_i) \notin \mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{B}))\&\Phi_{11}(\mathfrak{A}, \mathfrak{B}, y_1, \dots, y_k, z_1, \dots, z_k))\vee$$

$$(\bigvee_{\mathfrak{A}_1\subset\mathfrak{A}} ((\sum_{a_i\in(\mathfrak{A}\setminus\mathfrak{A}_1)} y_i) \in \mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{B})\&\Phi_{12}(\mathfrak{A}, \mathfrak{B}, \mathfrak{A}_1, y_1, \dots, y_k, z_1, \dots, z_k)))).$$

$\Phi_1(\mathfrak{A}, \mathfrak{B})$ tells us that either there is $u \in L(\mathfrak{A})$ such that $y_1 + \cdots + y_k$ is equal to $\overline{h(\mathfrak{A}, \mathfrak{B}) + u}$ in $L(\mathfrak{A}, \mathfrak{B})$ and then $[y_1 + \cdots + y_k] = [z_1 + \cdots + z_k]$ in $M(\mathfrak{A}, \mathfrak{B})$, or there is $\mathfrak{A}_1 \subset \mathfrak{A}$ such that $(\sum_{a_i\in(\mathfrak{A}\setminus\mathfrak{A}_1)} y_i) \in \mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{B})$ and $\Phi_{12}(\mathfrak{A}, \mathfrak{B}, \mathfrak{A}_1, y_1, \dots, y_k, z_1, \dots, z_k)$. The last formula will be described below.

The correctness of $\Phi_1(\mathfrak{A}, \mathfrak{B})$ is followed from (5) and

LEMMA 5. *If $x, y \in L(\mathfrak{A})$ and $\overline{x} = \overline{y}$, then for any $\mathfrak{A}_1 \subset \mathfrak{A}$, $\mathfrak{A}_1(x) \in \mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{B})$ iff $\mathfrak{A}_1(y) \in \mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{B})$.*

Proof. By contradiction. Suppose $\mathfrak{A}_1(x) \in \mathfrak{M}(\mathfrak{A}_1, \mathfrak{B})$. If $v \in L(\mathfrak{A}_1)$, $w \in L(\mathfrak{A})$, and $\overline{\mathfrak{A}_1(y) + v} = \overline{h(\mathfrak{A}, \mathfrak{B}) + w}$, then

$$\overline{\mathfrak{A}_1(x) + v + (\mathfrak{A} \setminus \mathfrak{A}_1)(x)} =$$

$$\overline{\mathfrak{A}_1(y) + v + (\mathfrak{A} \setminus \mathfrak{A}_1)(y)} = \overline{h(\mathfrak{A}, \mathfrak{B}) + w + (\mathfrak{A} \setminus \mathfrak{A}_1)(y)}.$$

It means that $\mathfrak{A}_1(x) \notin \mathfrak{N}(\mathfrak{A}_1, \mathfrak{B})$. A contradiction. If $u \in L(\mathfrak{A} \setminus \mathfrak{A}_1)$ and $(\mathfrak{A}_1(y) + u) \in \mathfrak{N}(\mathfrak{A}_1, \mathfrak{B})$, then $(\mathfrak{A}_1(x) + u) \in \mathfrak{N}(\mathfrak{A}_1, \mathfrak{B})$. A contradiction. If $\mathfrak{A}_1(x) \in \mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{B})$, then there exists $u \in L(\mathfrak{A} \setminus \mathfrak{A}_1)$ such that $(\mathfrak{A}_1(x) + u) \in \mathfrak{M}(\mathfrak{A}_1, \mathfrak{B})$. By the previous proof, $(\mathfrak{A}_1(y) + u) \in \mathfrak{M}(\mathfrak{A}_1, \mathfrak{B})$. Thus $\mathfrak{A}_1(y) \in \mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{B})$. ■

6 A construction of $\Phi_{12}(\mathfrak{A}, \mathfrak{B}, \mathfrak{A}_1, y_1, \dots, y_k, z_1, \dots, z_k)$

Let $\mathfrak{A}_1 \subset \mathfrak{A}$, $(\sum_{a_i\in(\mathfrak{A}\setminus\mathfrak{A}_1)} y_i) \in \mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{B})$, $\bigwedge_{i=1}^{k} G_{a_i}(y_i)$, $\bigwedge_{i=1}^{k} G_{a_i}(z_i)$, and $z_1 + \cdots + z_k = y_1 + \cdots + y_k$. Then $(\sum_{a_i\in(\mathfrak{A}\setminus\mathfrak{A}_1)} z_i) \in \mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{B})$ (see Lemma 5). Denote $z_1 + \cdots + z_k$ by z and $y_1 + \cdots + y_k$ by y. By definition, there exists $j \in \{1, \dots, i(\mathfrak{A}_1, \mathfrak{B})\}$ such that $\mathfrak{A}_1(y), \mathfrak{A}_1(z) \in \mathfrak{M}_{2,j}(\mathfrak{A}_1, \mathfrak{B})$. By definition, for any $\mathfrak{A}_3 \subset \mathfrak{A}_1$, $\mathfrak{A}_3((\mathfrak{A} \setminus \mathfrak{A}_1)(y)) \notin \mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{A}_3, \mathfrak{A}_1(y), \mathfrak{A}_1(z), \mathfrak{B})$.

Denote by

$$\mathfrak{R}(\mathfrak{A}_1(y), \mathfrak{A}_1(z), \mathfrak{B})$$

the set of all $u \in L(\mathfrak{A}_1)$ such that

$$\mathfrak{A}_3(u) \notin \mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{A}_3, \mathfrak{A}_1(y), \mathfrak{A}_1(z), \mathfrak{B})$$

for any $\mathfrak{A}_3 \subset \mathfrak{A}_1$. Denote by $\mathfrak{R}_1(\mathfrak{A}_1(y), \mathfrak{A}_1(z), \mathfrak{B})$ the set of all minimal elements from $\mathfrak{R}(\mathfrak{A}_1(y), \mathfrak{A}_1(z), \mathfrak{B})$. The set $\mathfrak{R}_1(\mathfrak{A}_1(y), \mathfrak{A}_1(z), \mathfrak{B})$ is finite (the theorem of Dickson, [1], p.129).

It followed from (6) that for any $u \in \mathfrak{R}_1(\mathfrak{A}_1(y), \mathfrak{A}_1(z), \mathfrak{B})$, there exists $v(u) \in L(\mathfrak{A}_1)$ such that $\overline{\mathfrak{A}_1(y) + u} = \overline{\mathfrak{A}_1(z) + v(u)}$ in $L(\mathfrak{A}, \mathfrak{B})$.

By definition, there exists $u \in \mathfrak{R}_1(\mathfrak{A}_1(y), \mathfrak{A}_1(z), \mathfrak{B})$ such that $u \leqslant (\mathfrak{A} \backslash \mathfrak{A}_1)(y)$.

It demonstrates that $\Phi_{12}(\mathfrak{A}, \mathfrak{B}, \mathfrak{A}_1, y_1, \ldots, y_k, z_1, \ldots, z_k)$ can be taken as

$$\bigvee_{u \in \mathfrak{R}_1(\mathfrak{A}_1(y), \mathfrak{A}_1(z), \mathfrak{B})} (\exists w)((\mathfrak{A} \setminus \mathfrak{A}_1)(y) = u + w \&$$

$$\Phi_{10}(\mathfrak{A}_1, \mathfrak{B}(\mathfrak{A}_1, \sum_{a_i \in (\mathfrak{A} \setminus \mathfrak{A}_1)} z_i), (\mathfrak{A} \setminus \mathfrak{A}_1)(z) + w + v(u), (\mathfrak{A} \setminus \mathfrak{A}_1)(z))).$$

Here we use the following shorting: we write d instead of $n_1 d_1, \ldots, n_s d_s$ for natural numbers $n_1, \ldots, n_s$ if the form d on the free generators $d_1, \ldots, d_s$ is $n_1 d_1 + \cdots + n_s d_s$.

7 Conclusion

There exists an algorithm that for any finite defining commutative semigroup $L(\mathfrak{A}', \mathfrak{B}')$, determines whether or not $\Phi(\mathfrak{A}, \mathfrak{B})$ is true in $L(\mathfrak{A}', \mathfrak{B}')$. Indeed, $\Phi(\mathfrak{A}, \mathfrak{B})$ is true in $L(\mathfrak{A}', \mathfrak{B}')$ iff $L(\mathfrak{A}', \mathfrak{B}')$ and $L(\mathfrak{A}, \mathfrak{B})$ are isomorphic. There exists an algorithm to decide whether or not $L(\mathfrak{A}', \mathfrak{B}')$ and $L(\mathfrak{A}, \mathfrak{B})$ are isomorphic (see [6]).

BIBLIOGRAPHY

[1] A.H. Clifford and G.B.Preston. *The algebraic theory of semigroups*, volume 2. American Math.Soc., Providence, R.I., 1967.

[2] M.A. Taitslin. On elementary theories of commutative semigroups. *Algebra i logika*, 5(4):55–89, 1966. Russian.

[3] M.A. Taitslin. On elementary theories of commutative semigroups with cancelation. *Algebra i logika*, 5(1):51–69, 1966. Russian.

[4] M.A. Taitslin. Two remarks on commutative semigroups isomorphism. *Algebra i logika*, 6(1):95–116, 1967. Russian.

[5] M.A. Taitslin. Algorithmic problem for commutative semigroups. *Sov.Math.Dokl.*, 9(1):201–204, 1968.

[6] M.A. Taitslin. The isomorphism problem for the commutative semigroups has the positive solution. In *Model theory and its applications*, pages 75–81. Kazakh State University, Alma–Ata, 1980. Russian.

[7] B.I. Zilber. On isomorphism and elementary equivalence of commutative semigroups. *Algebra and logic*, 9(6):667–671, 1970. Russian.