A GENERAL CONDITION FOR COLLAPSE RESULTS

MICHAEL A. TAITSLIN

To N.A.Shanin's 80-th Anniversary

ABSTRACT. In [1] the collapse result theorem was proved for locally generic queries over ordered domain with Pseudo-finite Homogeneity Property.

In a very interesting paper of Baldwin and Benedikt [2] the collapse result theorem was proved for locally generic queries over ordered domains without the independence property. It means that over such a domain, order-generic extended queries fail to express more than restricted queries.

It was observed by Baldwin and Benedikt in [2] that any theory without the independence property is P-reducible.

I prove that, for the P-reducible theories, a version of Pseudo-finite Homogeneity Property holds.

1. INTRODUCTION

Database relations (tables) are always going to be finite. However, it is often convenient to assume that there is an infinite *domain*—for example, the integer or rational numbers or the strings—such that the data elements are chosen from this domain. Functions and relations defined over the entire domain, like $<$ and $+$, may also be used in querying, for example, if the language of first-order logic FO is used as the query language, its formulas may use the database relations as well as the domain relations, while variables range over the entire domain.

More formally, a structure of a relational signature L is a non-empty set with a mapping that assigns to every relational symbol in L a relation of the same arity over the set. Let U be an infinite structure over the signature L. This structure is called the *universe.* In this paper, we always consider ordered universes. This means that L includes a binary relational symbol $<$ whose interpretation in U satisfies the axioms of linear order. Let us denote $L_0 = \{<\}$. A *database scheme SC* is a finite collection of relational symbols of fixed arities. A *database state* (over U) is an assignment to

Date: April 23, 2001.

1991 *Mathematics Subject Classification.* Primary 68P15; Secondary 03C40, 03C52.

Key words and phrases. local genericity, finite database state, collapse results.

The address for correspondence: 108 Gorkogo St., apt.148, Tver, 170005, Russia.

The work of the author was partially supported by the Russian Foundation of Basic Research (project code: 0001 00254).

these relational symbols of concrete relations of corresponding arities over U. These relations are called *database relations.* A database state is called a *finite database state* if all the relations are finite. We fix a database scheme SC and denote $L_0^+ = L_0 \bigcup SC$, and $L^+ = L \bigcup SC$.

A *database query* can formally be defined as a mapping that takes in a database state (of a fixed database scheme), and produces a new relation, of a fixed arity, over U. Thus, every query has an arity. Specifically, queries of arity 0 are called *Boolean queries.* A Boolean query defines a mapping from database states to $\{0, 1\}$, or, in other words, a set of database states of a given database scheme.

Queries can be formulated using *query languages,* the simplest being the language of first-order logic FO. Formulas (queries) of this language use $=$, as well as the relational symbols of the signature and of the database scheme. Thus, a database state essentially defines a structure of a larger signature with U as the domain; then a formula with n free variables defines an n-ary relation over U; sentences define Boolean queries. We say that two L^+-formulas with n free variables are equivalent over finite states over U if they define the same n-ary query.

We consider two languages for querying. Queries of the first one are FO formulas of the signature L_0^+—we call them *restricted.* Queries of the second language are FO formulas of the signature L^+—we call them *extended.*

Speaking informally, for querying, the restricted query language uses stored information only but the extended one also uses a general knowledge, for example, the knowledge of the addition of the natural numbers in the case when the universe is $(\mathbb{N}, <, +)$.

We will also use a notion of a *locally generic* query. It was proposed in [3]. A k-ary query Q is said to be *locally generic over finite states* if $\bar{a} \in Q(s)$ iff $f(\bar{a}) \in Q(f(s))$, for any partial $<$-isomorphism $f : X \to U$ with $X \subseteq U$, for any finite state s over X, and for any k-tuple $\bar{a}$ in X.

The problem we are interested in the paper is:

does the general knowledge improve the expressive power of the locally generic queries over finite states?

The problem was considered in a lot of papers. See [1] for the history and the bibliography. The answer is negative in many cases. Usually the theorem that the answer is negative for a universe is called the collapse result or the collapse theorem for the universe.

A sufficient condition for the collapse result was proposed in [1]. It is the Pseudo-finite Homogeneity Property. In [2] the collapse result was proved for universes without the independence property. It was proved in [2] that for any universe U without the independence property, there is a P-reducible for L small (M, I) such that $M \equiv U$.

A problem arises: what is the correlation between the properties.

I propose a deeper version of Theorem 5.1 from [1]. The version contains Theorem 5.2 from [1].

It was proved in [2] that for any universe U without the independence property, there is a P-reducible for L small (M, I) such that $M \equiv U$.

Using my version of Theorem 5.1, I prove that if for a universe U there is a P-reducible for L small (M, I) such that $M \equiv U$, then for any Boolean extended ρ-query ϕ which is locally generic for finite database states over U, there is a restricted ρ-query ψ which is equivalent to ϕ over finite database states over U.

Indeed, for universes without the independence property, a version of the Pseudo-finite Homogeneity Property holds.

We use the notation from [1].

For an arbitrary signature L, an L-theory is defined to be a set of first-order L-sentences (that is, formulas of signature L without free variables). For a class K of structures of an arbitrary signature L (in symbols, L-structures), the first-order L-theory of K (in symbols, $\mathrm{Th}(K)$) is defined to be the set of all first-order L-sentences which hold in every structure in K. Two L-structures M and N are called *elementarily equivalent* (in symbols, $M \equiv N$), if ϕ holds in M iff ϕ holds in N, for any L-sentence ϕ. An L-theory T is said to be complete if all its models are elementarily equivalent.

Let ρ be a database scheme $\{R_1, \ldots, R_n, c_1 \ldots, c_k\}$. We denote $L \cup \rho$ by $L(\rho)$. A ρ-state s over an L-structure W is said to be *pseudo-finite in* W if (W, s) is a model of the first-order $L(\rho)$-theory $F(W, \rho)$ of all (W, r), where r is a finite ρ-state over W.

For a first-order $L(\rho)$-sentence ψ and $m < \omega$, there is a first-order L-sentence ψ_m such that, for any L-structure V, the sentence ψ_m holds in V iff ψ holds for all ρ-states over V, whose active domain has cardinality at most m. Thus, $\psi \in F(W, \rho)$ iff $\{\psi_m : m < \omega\} \subseteq \mathrm{Th}(W)$. It follows that $W \equiv V$ implies $F(V, \rho) = F(W, \rho)$. For a complete L-theory T, the first-order $L(\rho)$-theory $F(T, \rho)$ is well-defined to be $F(W, \rho)$, where W is an arbitrary model of T.

Let ρ' be a disjoint copy $\{R'_1, \ldots, R'_n, c'_1 \ldots, c'_k\}$ of ρ. For an $L(\rho)$-sentence θ denote by $\theta(\rho')$ its $L(\rho')$-copy, that is, the result of replacement of every occurrence of R_i and c_j in θ with R'_i and c'_j, respectively. Let $\bar{\rho} = \rho \cup \rho'$.

As we will use the standard technique of so-called special models, we summarize its basic definitions and facts (see [4] for detail).

For a structure M of an arbitrary signature L and a subset A of M, denote by $L(A)$ the signature obtained by adjoining to L names for the elements of A. We do not normally distinguish between elements of A and their names.

We say that M is an *elementary substructure* of N (in symbols, $M \preceq N$ or $N \succeq M$), if M is a substructure of N, and ϕ holds in M iff ϕ holds in N, for any $L(M)$-sentence ϕ.

A set p of first-order $L(A)$-formulas with one free variable x is said to be a *type* over A in M if every finite subset $\{\phi_1(x), \ldots, \phi_k(x)\}$ of p is realized in M (that is, $(\exists x)(\phi_1(x) \& \ldots \& \phi_k(x))$ holds in M), and, for every $L(A)$-formula $\phi(x)$, either $\phi \in p$ or $\neg\phi \in p$. We say that a subset q of p *isolates* p if p is the only type over A in M containing q.

Let A be a subset of M. For any $N \succeq M$ and $a \in N$, the set of all $L(A)$-formulas $\phi(x)$ such that $\phi(a)$ holds in N forms a type over A in M; denote it by $\mathrm{tp}(a/A)$. For any type p over A in M, there are $N \succeq M$ and $a \in N$ such that $p = \mathrm{tp}(a/A)$. We denote $\mathrm{tp}(A)$ the set of all $L(A)$-sentences which hold in M.

For a cardinal λ, a structure M is said to be *λ-saturated* if any type p over any its subset A of power $< \lambda$ is realized in M; that is, $p = \mathrm{tp}(a/A)$, for some $a \in M$. For any infinite $\lambda \geqslant |L|$, every two elementarily equivalent λ-saturated structures of power λ are isomorphic. $|L|$ denotes the cardinality of L.

A structure M of power λ is called *special* if M is the union of a family $\{M_\mu : \mu \text{ is a cardinal } < \lambda\}$, where $M_\mu \preceq M_\nu \preceq M$ for $\mu < \nu < \lambda$, and each M_μ is μ^+-saturated. Here μ^+, as usual, denotes the least cardinal greater than μ. Every two elementarily equivalent special structures of the same power are isomorphic. For any infinite L-structure M and any cardinal $\lambda > |L|, |M|$ with $\lambda^* = \lambda$, there exists a special $N \succeq M$ of power λ. Here $\aleph_\alpha^*$ is defined to be $\sum_{\beta<\alpha} 2^{\aleph_\beta}$, and $|M|$ is the cardinality of M. It is easy to construct cardinals λ with $\lambda^* = \lambda$ of arbitrarily large cofinality.

2. Main results

Theorem 2.1. *For any universe U and any Boolean extended ρ-query ψ which is locally generic for finite database states over U, the following conditions are equivalent:*

1. *There is a restricted ρ-query ψ which is equivalent to ϕ over finite database states over U*
2. *There are a model $V \equiv U$ and an infinite indiscernible sequence I in V such that for any uncountable power κ with $\kappa = \kappa^*$, for any special $(W, J) \equiv (V, I)$ of power κ the query ϕ is generic over pseudo-finite states over J in W.*

Proof. (1)$\Rightarrow$(2). Suppose ϕ is equivalent to a restricted query ψ, for finite database states over U. Then $\phi \leftrightarrow \psi$ is in $F(U, \rho)$ and so in $F(V, \rho)$, for every $V \equiv U$. As ψ, being restricted, is generic even for all states over V, the genericity of ϕ for pseudo-finite states in V follows.

(2)$\Rightarrow$(1). Let $T = Th(V,\ I)$. First we show that (3) implies the inconsistency of the following set of formulas

$$\Gamma = (AD(\rho) \subseteq I) \cup (AD(\rho') \subseteq I) \cup \\ F(T, \bar\rho) \cup \{\theta(\rho) \leftrightarrow \theta(\rho') : \theta \text{ is an } L_0(\rho)\text{-sentence}\} \cup \{\phi(\rho), \neg\phi(\rho')\}.$$

Towards a contradiction, suppose $(W,\ J,\ r,\ r')$ is a model of Γ. We can assume that it is a special model and $|W| = \kappa$. Then its reducts W, $(W|L_0, r)$ and $(W|L_0, r')$ are special, too. Due to Γ, the models $(W|L_0, r)$ and $(W|L_0, r')$ are elementarily equivalent and hence isomorphic. In other words, there is an L_0-automorphism of W which transforms r to r'. Moreover, r and r' are states over J in W. As (r, r') is pseudo-finite in W, and $\phi(\rho)$ holds in (W, r) but $\phi(\rho')$ fails in (W, ρ'), the query ϕ is not generic for pseudo-finite states over J in W. Since $(V, I) \equiv (W, J)$, we have a contradiction with (2).

By compactness, there is a finite inconsistent part of Γ. Then, for some $m < \omega$ and $L_0(\rho)$-sentences $\theta_0, \dots, \theta_{m-1}$,

$$F(T, \bar{\rho}),\ \wedge_{i<m}(\theta_i(\rho) \leftrightarrow \theta_i(\rho')),\ \phi(\rho)\ \vdash\ \phi(\rho').$$

Indeed, the local genericity of ϕ implies consistency of

$$(AD(\rho) \subseteq I),\ (AD(\rho') \subseteq I),\ F(T, \bar{\rho}),\ \wedge_{i<m}(\theta_i(\rho) \leftrightarrow \theta_i(\rho')),\ \phi(\rho),\ \neg\phi(\rho')$$

if

$$F(T, \bar{\rho}),\ \wedge_{i<m}(\theta_i(\rho) \leftrightarrow \theta_i(\rho')),\ \phi(\rho),\ \neg\phi(\rho')$$

is consistent.

Denote $\theta^1 = \theta$, $\theta^0 = \neg\theta$. It easily follows that, for some $\tau_0, \dots, \tau_{l-1} \in 2^m$, the extended query ϕ is equivalent, for finite states over U, to the restricted query $\psi = \vee_{i<l} \wedge_{j<m} \theta_i^{\tau_{ij}}$. ∎

Let P be a unary relation name. An (L, P)-structure (M, I) is called *small* iff the following properties hold:

- I is a complete dense linearly ordered without endpoints subset of M
- $(I, <)$ is ω-saturated
- I is indiscernible sequence in M
- M is $|I|^+$-saturated

The main part of the definition is due to [2].

It is proved in [4] that for any infinite universe U, a small (M, I) with $M \equiv U$ does exist.

An (L, P)-structure (N, J) is called *pseudo-small* iff there exists a small (L, P)-structure (M, I) such that $(M, I) \equiv (N, J)$.

Theorem 2.2. *For any small (M, I), I is an indiscernible sequence in (M, I).*

Proof. Any local order monomorphism I in I can be extended to an order automorphism of I, and the automorphism can be extended to an automorphism of (M, I). ∎

Corollary 2.3. *For any pseudo-small (M, I), I is an indiscernible sequence in (M, I).*

By an *order* formula we mean a quantifier-free formula in order and equality. We write $\phi(\bar{x}, \bar{y})$ to denote that any free variable of formula ϕ appears in $\bar{x}, \bar{y}$.

Definition 2.1. *Any L-formula is P-bounded. If ϕ and ψ are P-bounded, then $(\phi \& \psi)$, $(\phi \vee \psi)$, $\neg\phi$, $(\forall x \in P)\phi$, and $(\exists x \in P)\phi$ are P-bounded.*

Definition 2.2. *A small (M, I) is called P-reducible for L (for (L, P)) iff*

- *for every L-formula (P-bounded (L,P)-formula) $\phi(\bar{x},\bar{y})$, there is an order formula $\psi(\bar{w},\bar{y})$ such that for every $\bar{m}$ there is a $\bar{c}_{\bar{m}} \in I$ such that*
$$(\forall \bar{y} \in P)(\psi(\bar{c}_{\bar{m}},\bar{y}) \equiv \phi(\bar{m},\bar{y})).$$
- *for every P-bounded (L,P)-formula ϕ the formula $(\exists x)\phi$ is equivalent in (M,I) to a P-bounded (L,P)-formula.*

Definition 2.3. *A formula $\phi(\bar{x},\bar{y})$ is called independent in M iff for any natural n there exist $\bar{a}_1,\ldots,\bar{a}_n$ in M such that for any $I \subseteq \{1,\ldots,n\}$, there exists $\bar{b}_I$ in M such that*
$$((M,\bar{a}_i,\bar{b}_I) \models \phi(\bar{a}_i,\bar{b}_I)) \quad \Leftrightarrow \quad i \in I.$$
A theory has the independence property iff the theory has a model with an independent formula.

Theorem 2.4 (Baldwin and Benedikt, [2])**.** *Any small model (M,I) where M is a model of a theory without the independence property is P-reducible for L.*

Theorem 2.5. *If a small model (M,I) is P-reducible for L, then the model is P-reducible for (L,P).*

Proof. Suppose that for $\phi(\bar{x},\bar{y},z)$, there is an order formula $\psi(\bar{w},\bar{y},z)$ such that for every $\bar{m}$ there is a $\bar{c}_{\bar{m}} \in I$ such that
$$(\forall \bar{y} \in P)(\forall z \in P)(\psi(\bar{c}_{\bar{m}},\bar{y},z) \equiv \phi(\bar{m},\bar{y},z)).$$
Then
$$(\forall \bar{y} \in P)((\forall z \in P)\psi(\bar{c}_{\bar{m}},\bar{y},z) \equiv (\forall z \in P)\phi(\bar{m},\bar{y},z)).$$
∎

Theorem 2.6. *If for a universe U there is a P-reducible for L small (M,I) such that $M \equiv U$, then for any Boolean extended ρ-query ϕ which is locally generic for finite database states over U, there is a restricted ρ-query ψ which is equivalent to ϕ over finite database states over U.*

Proof. We use theorem 2.1. Let (M,I) be a P-reducible model, κ be a uncountable power with $\kappa = \kappa^*$, $M \equiv U$, ϕ be a Boolean extended ρ-query, which is locally generic for finite database states over U, and let (N,J) be a special model of power κ such that $(N,J) \equiv (M,I)$. We have to prove that ϕ is generic over pseudo-finite states over J in N.

Let s and s' be pseudo-finite states over J in N, and let g be an automorphism of N such that $g(s) = s'$. We have to prove that
$$(N,s) \models \phi \Leftrightarrow (N,s') \models \phi.$$
It suffices to prove that Duplicator has a winning strategy in the Fraïssé-Ehrenfeucht n-game for any number n of the steps of the game.

Note that g may not keep J. But the restriction of g to the active domain of s can be extended to an automorphism h of $(J,<)$.

Indeed, any special model is ω-saturated. So $(J,<)$ is ω-saturated. Therefore for any $u<v$ from J, there is an automorphism $h_{u,v}$ of $(J,<)$ such that $h_{u,v}(u)=g(u)$ and $h_{u,v}(v)=g(v)$. For any $u<v$ from J, we fix such an automorphism $h_{u,v}$.

The active domain of s is pseudo-finite (Lemma 5.5 from [1]). So for any $x \in (J \setminus AD(s))$ there are u and v in $AD(s)$ such that either u is minimal in $AD(s)$, v is maximal in $AD(s)$, and either $x<u$ or $v<x$, or $u<x<v$, there is no element of $AD(s)$ between u and x, and there is no element of $AD(s)$ between x and v. We put $h(x)=h_{u,v}(x)$.

For $j \leqslant n$, we define finite sequences $\bar{a}$ and $\bar{a}'$ of elements of $N \setminus J$, and a natural number k_j by induction. By the definition, k_j is the length (the number of all the elements of the sequence) of $\bar{a}$ after the step j. The length of $\bar{a}'$ is equal to the length of $\bar{a}$. Let k_0 be 0.

For $i=1,\dots,k_j$, let $h_j(a_i)={a'}_i$, and let $h_j(i)=h(i)$ for any $i \in J$. So h_j maps $J \cup \{a_1,\dots,a_{k_j}\}$ onto $J \cup \{{a'}_1,\dots,{a'}_{k_j}\}$. Suppose h_j preserves any P-bounded (L,P)-formula.

Suppose Spoiler starts a new round $j+1$ of the game and chooses an element $b_{j+1} \in (N,s)$ (or ${b'}_{j+1} \in (N,s')$).

If the chosen element belongs to J, we put $k_{j+1}=k_j$. In this case we put ${b'}_{j+1}=h(b_{j+1})$ (or we put $b_{j+1}=h^{-1}({b'}_{j+1})$).

Suppose the chosen element does not belong to J. Consider the case Spoiler chooses an element in (N,s). Another case is considered similarly.

We put $k_{j+1}=k_j+1$ and $a_{k_{j+1}}=b_{j+1}$.

We have to choose ${a'}_{k_{j+1}} \notin J$ such that h_{j+1} preserves any P-bounded (L,P)-formula.

For any P-bounded (L,P)-formula $\psi(\bar{x},\bar{y})$ the free variables of which are in $\bar{x},\bar{y}$ where $\bar{x}$ is a k_{j+1}-tuple of variables, we choose an order formula $\theta_\psi(\bar{w}_\psi,\bar{y})$ and a tuple $\bar{c}_\psi$ of elements of J such that

$$(\forall \bar{y} \in P)(\theta_\psi(\bar{c}_\psi,\bar{y}) \equiv \psi(\bar{a},\bar{y}))$$

holds in (N,J).

The set of all the elements of all the chosen tuples is countable. It follows from the definition of (N,J) that the structure is $\aleph_1$-saturated. Thus there is ${a'}_{k_{j+1}} \notin J$ such that

$$(\forall \bar{y} \in P)(\theta_\psi(h(\bar{c}_\psi),\bar{y}) \equiv \psi(\bar{a}',\bar{y}))$$

holds in (N,J) for any ψ. We are done. ∎

Corollary 2.7. *If U lacks the independence property then for any extended query ϕ which is locally generic for finite database states over U, there is a restricted query ψ which is equivalent to ϕ over finite database states over U.*

Proof. It follows from Theorem 2.6 and Theorem 5.2 from [2] ∎

3. Generalizations

Definition 3.1. *Let M be a model of T and an infinite set I be an indiscernible sequence in M. A complete L-theory T has the (M,I)-Pseudo-finite Homogeneity Property iff whenever $(N,J) \equiv (M,I)$, A and B are pseudo-finite subsets of J in the model (N,J), C and D are finite subsets of $N \setminus J$, and $h : (A \cup C) \to (B \cup D)$ is an elementary map in (N,J) with ω-saturated (N,J,A,B,h), for any $a \in N$ there is $b \in N$ such that $h \cup \{(a,b)\}$ is an elementary map in (N,J).*

Theorem 3.1. *Let M be a model of T and an infinite set I be an indiscernible sequence in M. Suppose the first-order theory of a universe U has the (M,I)-Pseudo-finite Homogeneity Property. Let an extended query ϕ be locally generic over finite states over U. Then ϕ is equivalent over finite states over U to a restricted query.*

The proof of theorem 3.1 is the same as the proof of Theorem 5.4 in [1].

Definition 3.2. *Let M be a model of T and an infinite set I be an indiscernible sequence in M. A complete theory T has the (M,I)-Isolation Property iff for any special $(N,J) \equiv (M,I)$, for any pseudo-finite subset A of J, any finite subset C of $N \setminus J$, and any element a of N, there is a countable $A_0 \subseteq A$ such that* $\mathrm{tp}(a/(A_0 \cup C))$ *isolates* $\mathrm{tp}(a/(A \cup C))$ *in (N,J).*

Theorem 3.2. *The (M,I)-Isolation Property implies the (M,I)-Pseudo-finite Homogeneity Property.*

The proof of theorem 3.2 is the same as the proof of Theorem 5.8 in [1].

Theorem 3.3. *Let (M,I) be a P-reducible model and M be a model of a complete theory T without the independence property. Then T has (M,I)-Isolation Property.*

The proof of Theorem 2.6 proves Theorem 3.3.

References

[1] O.V. Belegradek, A.P. Stolboushkin, and M.A. Taitslin. Extended order-generic queries. *Annals of Pure and Applied Logic*, 97(1–3):85–125, 1999.
We consider relational databases organized over an ordered domain with some additional relations—a typical example is the ordered domain of rational numbers together with the operation of addition. In the focus of our study are the first-order (FO) queries that are invariant under order-preserving "permutations"—such queries are called order-generic. It has recently been discovered that for some domains order-generic FO queries fail to express more than pure order queries. For example, every order-generic FO query over rational numbers with + can be rewritten without +. For some other domains, however, this is not the case.
We provide very general conditions on the FO theory of the domain that ensure the collapse of order-generic extended FO queries to pure order queries over this domain: the *Pseudo-finite Homogeneity Property* and a stronger *Isolation Property*. We further distinguish one broad class of domains satisfying the Isolation Property, the so-called *quasi-o-minimal* domains. This class includes all the o-minimal domains, but also the ordered group of integer numbers and the ordered semigroup of natural numbers, and some other domains.

An important difference of this paper from the recent series of related papers is that we generalize all the notions to the case of finitely representable database states—as opposed to finite states—and develop a general lifting technique that, essentially, allows us to extend any result of the kind we are interested in, from finite to finitely-representable states. We show, however, that these results cannot be transfered to arbitrary infinite states.
[2] J.T. Baldwin and M. Benedikt. Stability theory, permutations of indiscernibles, and embedded finite models. Preprint, 1998.
[3] M. Benedikt, G. Dong, L. Libkin, and L. Wong. Relational expressive power of constraint query languages. In *Proc. 15th ACM Symp. on Principles of Database Systems*, pages 5–16, 1996.
[4] C.C. Chang and H.J. Keisler. *Model Theory.* North Holland, 3rd edition, 1990.

M.A.Taitslin, Department of Computer Science, Tver State University, Tver, 170000, Russia
E-mail address: `Michael.Taitslin@tversu.ru`